Ordered rate constitutive theories in Lagrangian description for thermoelastic solids and thermoviscoelastic solids with and without memory using Helmholtz free energy density

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Abstract

The research work presented here considers development of constitutive theories in Lagrangian description for homogeneous, isotropic, compressible and incompressible thermoelastic solids, thermoviscoelastic solids without memory, and thermoviscoelastic solids with memory. Since conservation of mass, balance of momenta, and the first law of thermodynamics assume the existence of a stress field and heat vector without regard to how they are arrived at, the constitutive theories for the stress field and heat vector must be derived using the second law of thermodynamics to ensure thermodynamic equilibrium in the deforming matter during evolution. In the present work, we use the entropy inequality resulting from the second law of thermodynamics expressed in terms of the Helmholtz free energy density Φ. The initial choice of dependent variables is directly from the entropy inequality: Φ, second Piola-Kirchhoff stress tensor $\boldsymbol{\sigma}^{[0]}$, entropy density η, and heat vector $\boldsymbol{q}$. The argument tensors are established based on desired physics, *i.e.*, choices are made depending on whether the solid matter is thermoelastic, thermoviscoelastic without memory, or thermoviscoelastic with memory. The use of the entropy inequality with the desired choices of argument tensors of the dependent variables allows us to determine the final choice of dependent variables in the constitutive theories as Φ, $\boldsymbol{\sigma}^{[0]}$, and $\boldsymbol{q}$ as well as their argument tensors (depending on the desired physics).

In the case of thermoelastic solids, the entropy inequality provides conditions from which the constitutive theories for $\boldsymbol{\sigma}^{[0]}$ and $\boldsymbol{q}$ can be derived. For ther-

moviscoelastic solids with and without memory, the conditions resulting from the entropy inequality do not permit derivation of a constitutive theory for $\boldsymbol{\sigma}^{[0]}$. Using a decomposition of $\boldsymbol{\sigma}^{[0]}$ into ${}_e\boldsymbol{\sigma}^{[0]}$ and ${}_d\boldsymbol{\sigma}^{[0]}$ (equilibrium and deviatoric second Piola-Kirchhoff stress tensors), the constitutive theories for ${}_e\boldsymbol{\sigma}^{[0]}$ can be derived using the conditions resulting from the entropy inequality. However, the entropy inequality provides no mechanism to derive constitutive theories for ${}_d\boldsymbol{\sigma}^{[0]}$. In the present work, we use the theory of generators and invariants to derive constitutive theories for ${}_d\boldsymbol{\sigma}^{[0]}$. The constitutive theories for $\boldsymbol{q}$ consistent with $\boldsymbol{\sigma}^{[0]}$ or ${}_d\boldsymbol{\sigma}^{[0]}$ in the sense of argument tensors are also derived using the theory of generators and invariants. It is shown that for thermoelastic solids, the constitutive theory for $\boldsymbol{\sigma}^{[0]}$ and $\boldsymbol{q}$ are of rate zero in the Green's strain tensor $\boldsymbol{\varepsilon}$ ($\boldsymbol{\varepsilon}_{[0]}$), the constitutive theories for thermoviscoelastic solids without memory for ${}_d\boldsymbol{\sigma}^{[0]}$ and $\boldsymbol{q}$ are up to orders n in the Green's strain tensor, and the constitutive theories for thermoviscoelastic solids with memory for ${}_d\boldsymbol{\sigma}^{[0]}$ and $\boldsymbol{q}$ are of orders m and n in the second Piola-Kirchhoff stress tensor and Green's strain tensor.

Many simplified forms of the rate theories are presented and compared with those used currently to demonstrate the merits of this research and severe limitations and shortcomings of the constitutive theories used currently for the type of solids considered here. Numerical studies are also presented using the theories presented here, and in some cases, the results are compared with the currently used theories. These rate theories provide a comprehensive framework of constitutive theories that permit description of complex material physics.

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Chapter 1

Introduction, literature review, and scope of work

The constitutive theories describing the constitution of deforming matter is one of the most important aspects of the mathematical models of deforming matter. To ensure thermodynamic equilibrium in the deforming matter, the mathematical models must be derived using conservation laws. Conservation of mass, balance of momenta, and the first law of thermodynamics are independent of the constitution of the matter, thus the constitutive behavior of deforming matter must be established using the second law of thermodynamics. In the following, we present a brief literature review of the evolution of the subject of constitutive theories.

1.1 Literature review

General approach in constitutive theories

Coleman and Noll [4] provided a thermodynamic basis for the development of constitutive relationships by showing Helmholtz free energy density in terms of thermodynamic pressure for fluids and principal stretches in general solids. Pipkin and Rivlin [5] began development of a general constitutive theory framework to describe tensor properties of materials, to be applicable beyond just the field of continuum mechanics, focusing primarily on consequences of material symmetry and invariance, and explicitly eschewing assumptions of linearity as previous work had used. Rivlin further developed this work in a second paper [6].

The topic of nonlinear viscoelasticity, first studied by Rivlin [7], has only come into considerable study recently. A recent paper by Wineman [8] provides a good review on the current state of the topic. Most work in viscoelasticity uses predominantly integral equations, and therefore recent papers by Rajagopal and Saccomandi [9], Drapaca, Sivaloganathan, and Tenti [10], and Pucci and Saccomandi [11] are of note, due to their use of a rate theory approach. Other rate theory papers by Rajagopal [12] and Bulíček, Málek, and Rajagopal [13] claim to develop nonlinear theories of viscoelasticity, but rather than being developed from continuum mechanics principles, they purport to introduce nonlinearity in terms of a nonlinear relationship between linearized stress and strain, based on a modification of simple linear viscoelastic models, and as such, they lack merit.

Theory of generators and invariants

Starting with the 1955 paper by Rivlin and Ericksen [14], extensive work has been done to develop a representation theorem for scalar-, vector-, and tensor-valued functions for use in continuum mechanics. The development of this theory got its biggest start with Rivlin's followup [15] to his previous paper, in which he used the Hamilton-Cayley theorem to show reducibility of matrix polynomials. Subsequent papers by Spencer and Rivlin [16–19] and by Spencer [20,21] further refined an integrity basis for matrix polynomials. The topic was eventually taken over and developed extensively by Smith [22–25] and Wang [26–30]. A complete and minimal integrity basis for isotropic scalar, vector, symmetric tensor, and skew-symmetric tensor functions can be found in a review by Pennisi and Trovato [31].

Rate theories and nonlinear viscoelasticity

Though the first use of the concept of a rate of stress as a part of constitutive theory was due to Maxwell [32], in his efforts to develop a molecular theory describing the behavior of gases, it was not until Zaremba [33, 34] that a frame-invariant, convected stress rate measure that transforms according to tensor laws was introduced.

Up until the early 1950s, the fields of continuum mechanics and rheology were disjoint – the former being largely viewed as a theory for inviscid and linear fluids and linearly elastic solids, and the latter being concerned with developing mathematical theories to describe the behavior of the large number of materials for which the linear continuum mechanics theories did not provide adequate results. The rheological theories in the time, however, were largely either one-dimensional, or only valid for infinitesimal deformation. Beginning around 1955, however, substantial advances were made to unify the two theories.

Cotter and Rivlin [35] were among the first to explore a time-dependence of stress from a continuum mechanics perspective, and they provided general results that showed that the frame invariance and material isotropy assumptions in constitutive theory necessitated that any theory that includes convected stress rates of order m must also include strain rates of order $n \geq m$. Rivlin and Ericksen [14] provided a proof confirming an earlier result due to Oldroyd [36] regarding the selection of arguments for frame-invariant constitutive relations, which included strain rates but did not include consideration of stress rates.

Rivlin first explored stress relaxation behavior from a continuum mechanics perspective in [37], but did so by proposing that the then-known elastic constitutive theory include time as an independent variable, without a thorough development of the theory. Green and Rivlin [38] then developed a constitutive theory for materials exhibiting memory behavior, defining stress as an integrated time history of the deformation of the material. So called "hereditary" relationships have since been the focus of the vast majority of subsequent work on viscoelasticity.

Noll, in a comprehensive review [39], pointed out a means by which the hereditary relationships proposed by Green and Rivlin may be altered to eliminate the use of integral equations, thus relating these equations to the theories proposed by Rivlin and Ericksen, which describe what Noll calls materials of "differential type," and which we refer to hereafter as materials "without memory," and the theories proposed by Cotter and Rivlin, which Noll terms materials of "rate type," and which we call materials "with memory."

Green and Rivlin, along with Spencer, continued their development of theories for materials with memory [40] by incorporating new results from the theory of generators and invariants, relaxing some assumptions made in their previous work. The relations developed in this paper were still exclusively of the hereditary type. This theory is further compared to known theories from classical elasticity in a later paper by Rivlin [41]. In a third paper [42], Green and Rivlin incorporated results from Coleman and Noll [4] and those from the earlier paper by Rivlin and Ericksen [14] in their theory of materials with memory. Rivlin [6] also generalized these results to constitutive theories that are not necessarily restricted to the field of continuum mechanics, continuing along the same lines as in his previous paper [5] with Pipkin. Coleman later provided a paper [43] presenting a thermodynamic approach to materials with memory.

Linear viscoelasticity

Though the classical theory of linear viscoelasticity began in the late nineteenth century with Boltzmann [44], little work on the topic was done until the middle of the twentieth century. Coleman and Noll [45] gave the theory a solid foundation based on nonlinear continuum mechanics, including a linear theory based on finite deformation and specifically avoiding the use of the common assumption of a spring-and-dashpot network. A lengthy treatise [46] by Gurtin and Sternberg provides a good exposition on the topic and provides several new results. The most important of these include (i) making a distinction between materials of "relaxation type" and materials of "creep type", (ii) showing that linear isotropic viscoelastic materials, like their elastic counterparts, depend on only two (in this case, time-dependent) material parameters, necessitating the decomposition of the stress and strain tensors into equilibrium and deviatoric parts, (iii) determining the conditions for which relaxation type materials are also creep type materials, and (iv) relating differential and rate theories to the more commonly used integral theories.

Additional results were shown in the following few years. Herrera and Gurtin [47] provided important results concerning propagation of acceleration waves in materials of the relaxation type, which was generalized by Fisher and Gurtin [48] to include shock waves and waves of any order, and an important thermodynamic constraint on materials of relaxation type was shown by Gurtin and Herrera [49].

In 1973, Leitman and Fisher published an extensive review [50] of the work done up to that point on linear viscoelasticity, though it only covers linear theory based on infinitesimal deformation. Another useful review is found in the introduction of the report by Chin [51], particularly in applying the constraints due to Gurtin and Herrera [49] to differential or rate theories.

Most work since has primarily addressed integral theories of linear viscoelasticity, which we largely do not consider here. Some work, however, regarding rate theories based on linear viscoelasticity is still being done – recent examples include Rajagopal [12] and Bulíček, Málek, and Rajagopal [13]. In both of these works, however, the decomposition shown by Gurtin and Sternberg to be necessary for the material to be isotropic is completely neglected, yielding an incomplete basis for their theory.

It is perhaps fitting not to critique the published work in view of our present knowledge of the subject as these works are publications related to the evolution of the development of the subject of constitutive theories. However, in light of the maturity of the subjects of continuum mechanics and field theories, at this stage we must undertake a rigorous approach in the derivations of the constitutive theories that is based on conservation laws, principles of thermodynamics, and axioms and principles of constitutive theory in continuum mechanics. Such an approach requires that all constitutive theories be derived using the second law of thermodynamics, if the deforming matter is to be in thermodynamic equilibrium during evolution. In this process, we must ensure that the other three conservation laws are not violated. A systematic description of the research work undertaken here and the approach used to derive the constitutive theories for thermoelastic solids and thermoviscoelastic solids with and without memory has been described in the abstract and is not repeated here.

Chapter 2

Rate constitutive theories of order zero in Lagrangian description for thermoelastic solids

2.1 Introduction

For homogeneous, isotropic elastic solid matter undergoing finite deformation, the conservation laws must be satisfied by the evolution of the deformation process if it is to be in thermodynamic equilibrium [1–3]. Since the conservation of mass, balance of momenta, and balance of energy are independent of the constitution of the matter, the second law of thermodynamics or entropy inequality alone (if possible) must provide a mechanism for deriving the constitutive theories for the deforming matter [1–3]. The entropy inequality expressed in terms of Helmholtz free energy density Φ yields Φ, entropy density η, heat vector $\boldsymbol{q}$, and stress tensor $\boldsymbol{\sigma}^{[0]}$ (material derivative of order zero of second Piola-Kirchhoff stress tensor) as possible dependent variables in describing the constitution of the deforming solid matter.

If we choose second Piola-Kirchhoff stress tensor $\boldsymbol{\sigma}^{[0]}$ and Green's strain tensor $\boldsymbol{\varepsilon}$ as a conjugate pair, and if we define the material derivative of $\boldsymbol{\varepsilon}$ of order n as $\boldsymbol{\varepsilon}_{[n]}$, which in Lagrangian description is $\boldsymbol{\varepsilon}_{[n]} = \frac{D^n \boldsymbol{\varepsilon}}{Dt^n} = \frac{\partial^n \boldsymbol{\varepsilon}}{\partial t^n}$, then the strain tensor $\boldsymbol{\varepsilon}$ is in fact $\boldsymbol{\varepsilon}_{[0]}$, material derivative of $\boldsymbol{\varepsilon}$ of order zero. In the work presented here, we use $\boldsymbol{\varepsilon}_{[0]}$ instead of $\boldsymbol{\varepsilon}$: (i) to emphasize the fact that even for the simplest possible solid matter considered in this chapter, the constitutive theories for the stress tensor are in fact rate theories of order zero in both stress and strain tensors. (ii) to provide transparency of notations and derivation between this work and the following chapters that present higher order rate theories in Lagrangian description for solid matter with dissipation and with dissipation as well as memory.

If we choose $\boldsymbol{\varepsilon}_{[0]}$, material derivative of order zero of Green's strain tensor (*i.e.*, $\boldsymbol{\varepsilon}$), heat vector $\boldsymbol{g}$, and temperature θ as argument tensors of the dependent variables Φ, η, $\boldsymbol{\sigma}^{[0]}$, and $\boldsymbol{q}$, then the material derivative $\dot{\Phi}$ of Φ can be obtained using chain rule of differentiation.

When $\dot{\Phi}$ is substituted into the entropy inequality, the resulting conditions: (i) rule out η as a dependent variable in the constitutive theory; (ii) show that Φ is not a function of $\boldsymbol{g}$; (iii) establish that $\boldsymbol{\sigma}^{[0]}$ is deterministic if Φ is defined as a function of $\boldsymbol{\varepsilon}_{[0]}$ and θ; and (iv) also provide an inequality from which the constitutive theory for heat vector $\boldsymbol{q}$ can be derived. In this approach we have used $\boldsymbol{\sigma}^{[0]}$ and $\boldsymbol{\varepsilon}_{[0]}$ as a conjugate pair. The constitution of homogeneous, isotropic thermoelastic solid matter experiencing finite deformation is described in terms of constitutive theory for $\boldsymbol{\sigma}^{[0]}$ and $\boldsymbol{q}$.

This approach of deriving constitutive theories using the conditions resulting from the entropy inequality is rather straightforward in principle, but there are many details in the derivations of the constitutive theories and determination of material coefficients that require careful consideration. In the work presented here, it is shown that, using the conditions resulting from the entropy inequality, the constitutive theory for $\boldsymbol{\sigma}^{[0]}$ can be derived using three approaches: (i) assuming Helmholtz free energy density to be a function of the invariants of $\boldsymbol{\varepsilon}_{[0]}$ and θ and then using the condition resulting from the entropy inequality; (ii) using the theory of generators and invariants; and (iii) expanding Helmholtz free energy density in $\boldsymbol{\varepsilon}_{[0]}$ using Taylor series and then using the condition resulting from the entropy inequality. These three forms of the constitutive theories are examined for equivalence between them as well as their merits and shortcomings. The constitutive theory for the heat vector can also be derived using three different approaches: (i) strictly using the condition resulting from the entropy inequality; (ii) using theory of generators and invariants by assuming that the argument tensors of the heat vector $\boldsymbol{q}$ are $\boldsymbol{g}$ and θ; and (iii) using theory of generators and invariants by assuming that the argument tensors of $\boldsymbol{q}$ are $\boldsymbol{\varepsilon}_{[0]}$, $\boldsymbol{g}$, and θ. These constitutive theories for $\boldsymbol{q}$ resulting from the three approaches are also compared for equivalence, and their merits as well as shortcomings are also discussed.

2.2 Entropy inequality in Helmholtz free energy density Φ

Consider the second law of thermodynamics, *i.e.*, entropy inequality in Lagrangian description expressed in terms of Helmholtz free energy density Φ [1–3], and conjugate pair $\boldsymbol{\sigma}^*$ and $\dot{\boldsymbol{J}}$.

$$\rho_0\dot{\Phi} + \eta\dot{\theta} + \frac{|\boldsymbol{J}|q_i g_i}{\theta} - \sigma^*_{ki}\dot{\boldsymbol{J}}_{ik} \leq 0 \tag{2.1}$$

where ρ_0 is the density in the undeformed configuration (also used as reference configuration, *i.e.*, the configuration at the commencement of the evolution), $\boldsymbol{J}$ is the Jacobian of deformation, and $\boldsymbol{\sigma}^*$ is the first Piola-Kirchhoff stress tensor. A dot on all quantities refers to their material derivatives. From the balance of momenta and the first law of thermodynamics, we note that $\boldsymbol{\sigma}^*$ and $\boldsymbol{q}$ must be considered as dependent variables in the constitutive theory, as these depend on the constitution of the matter. The entropy inequality (2.1) confirms this, as these appear in (2.1) as well. Secondly, in addition to these two, η and Φ must also be considered as dependent variables in the constitutive theories. The temperature gradient $\boldsymbol{g}$ is deterministic from θ, which is self observable, hence $\boldsymbol{g}$ cannot be a dependent variable in the constitutive theories. Thus, at the outset, we have Φ, η, $\boldsymbol{\sigma}^*$ and $\boldsymbol{q}$ as dependent variables in the constitutive theories. Based on the principle of equipresence [1–3], we consider all possible measures of deformation as arguments of all four dependent variables in the constitutive theories. The Jacobian of deformation $\boldsymbol{J}$ is a fundamental measure of the deformation and hence must be an argument of all dependent variables. The temperature θ is a natural choice for thermoelastic behavior. In addition, the temperature gradient is also considered as an argument of all dependent variables. This choice is essentially necessitated by the heat vector $\boldsymbol{q}$, but we generalize this to include $\boldsymbol{g}$ as argument of all dependent variables. Thus, $\boldsymbol{J}$, $\boldsymbol{g}$, and θ are possible arguments of the dependent variables Φ, η, $\boldsymbol{\sigma}^*$, and $\boldsymbol{q}$ at the onset of the development of the constitutive theories. Some of the dependent variables and/or their arguments may be ruled out at a later stage if some restrictions in the development of the constitutive theories warrant so. Thus, at this stage, we have

$$
\begin{aligned}
\Phi &= \Phi(\boldsymbol{J}, \boldsymbol{g}, \theta) \\
\eta &= \eta(\boldsymbol{J}, \boldsymbol{g}, \theta) \\
\boldsymbol{\sigma}^* &= \boldsymbol{\sigma}^*(\boldsymbol{J}, \boldsymbol{g}, \theta) \\
\boldsymbol{q} &= \boldsymbol{q}(\boldsymbol{J}, \boldsymbol{g}, \theta)
\end{aligned}
\tag{2.2}
$$

Due to the principle of frame invariance, the rotation part in $\boldsymbol{J}$ cannot be part of the constitutive theory. Consider the polar decomposition of $\boldsymbol{J}$

$$
\boldsymbol{J} = \boldsymbol{R}\boldsymbol{S}_r = \boldsymbol{S}_l\boldsymbol{R}. \tag{2.3}
$$

In (2.3), $\boldsymbol{S}_r$ and $\boldsymbol{S}_l$ are the right and left stretch tensors that are symmetric and positive definite, and $\boldsymbol{R}$ is orthogonal, hence defines rotation and therefore cannot be part of the constitutive theory. Thus in (2.2), we must replace $\boldsymbol{J}$ with $\boldsymbol{S}_r$ (or $\boldsymbol{S}_l$ if so desired). However, $\boldsymbol{S}_r$ can be expressed in terms of Green's strain tensor $\boldsymbol{\varepsilon}$ or $\boldsymbol{\varepsilon}_{[0]}$

$$
\boldsymbol{S}_r^2 = (\boldsymbol{I} + 2\boldsymbol{\varepsilon}_{[0]}) \tag{2.4}
$$

Thus, we can further replace $\boldsymbol{S}_r$ by $\boldsymbol{\varepsilon}_{[0]}$, and we have the following from (2.2)

$$\begin{aligned}\Phi &= \Phi\big(\boldsymbol{\varepsilon}_{[0]}, \boldsymbol{g}, \theta\big)\\ \eta &= \eta\big(\boldsymbol{\varepsilon}_{[0]}, \boldsymbol{g}, \theta\big)\\ \boldsymbol{\sigma}^* &= \boldsymbol{\sigma}^*\big(\boldsymbol{\varepsilon}_{[0]}, \boldsymbol{g}, \theta\big)\\ \boldsymbol{q} &= \boldsymbol{q}\big(\boldsymbol{\varepsilon}_{[0]}, \boldsymbol{g}, \theta\big)\end{aligned} \tag{2.5}$$

In (2.2), we note that $\boldsymbol{g}$ and θ are tensors of rank one and zero, but $\boldsymbol{J}$ is not a tensor. In (2.5), the dependent variables as well as their arguments are all tensors. This suggests that final choices of arguments in (2.5) is admissible from the point of view that these are all tensors of various ranks. Before we proceed further, we express the last term in the entropy inequality in terms of conjugate pair $\boldsymbol{\sigma}^{[0]}$ and $\dot{\boldsymbol{\varepsilon}}_{[0]}$ or $\boldsymbol{\varepsilon}_{[1]}$ (see references [1–3] for details) instead of $\boldsymbol{\sigma}^*$ and $\boldsymbol{J}$.

$$\rho_0(\dot{\Phi} + \eta\dot{\theta}) + \frac{|J|\boldsymbol{q}^T\boldsymbol{g}}{\theta} - \sigma_{ki}^{[0]}(\varepsilon_{[1]})_{ik} \leq 0 \tag{2.6}$$

The form of entropy inequality (2.6) with (2.5) is suitable to proceed further in the development of the constitutive theories.

Using (2.5), we can obtain a more explicit form of $\dot{\Phi}$ using the chain rule of differentiation

$$\dot{\Phi} = \frac{\partial\Phi}{\partial(\varepsilon_{[0]})_{ki}}(\varepsilon_{[1]})_{ik} + \frac{\partial\Phi}{\partial g_i}\dot{g}_i + \frac{\partial\Phi}{\partial\theta}\dot{\theta} \tag{2.7}$$

Substituting from (2.7) in (2.6) and regrouping terms

$$\left(\rho_0\frac{\partial\Phi}{\partial(\varepsilon_{[0]})_{ik}} - \sigma_{ki}^{[0]}\right)(\varepsilon_{[1]})_{ik} + \rho_0\frac{\partial\Phi}{\partial g_i}\dot{g}_i + \rho_0\left(\frac{\partial\Phi}{\partial\theta} + \eta\right)\dot{\theta} + \frac{|J|q_i g_i}{\theta} \leq 0 \tag{2.8}$$

Inequality (2.8) is a polynomial of degree one in $\boldsymbol{\varepsilon}_{[1]}$, $\dot{\boldsymbol{g}}$, and $\dot{\theta}$. Since (2.8) must hold for all arbitrary but admissible choices of $\boldsymbol{\varepsilon}_{[1]}$, $\dot{\boldsymbol{g}}$, and $\dot{\theta}$, this is possible if the following conditions hold

$$
\begin{aligned}
\rho_0 \frac{\partial \Phi}{\partial (\varepsilon_{[0]})_{ik}} - \sigma_{ki}^{[0]} = 0; \quad \rho_0 \frac{\partial \Phi}{\partial \boldsymbol{g}} = \boldsymbol{0}; \\
\rho_0 \left(\frac{\partial \Phi}{\partial \theta} + \eta \right) = 0; \quad \frac{|J| q_i g_i}{\theta} \leq 0
\end{aligned}
\tag{2.9}
$$

Since $\rho_0 > 0$, $|J| > 0$, and $\theta > 0$, we can write (2.9) as

$$
\begin{aligned}
\rho_0 \frac{\partial \Phi}{\partial (\varepsilon_{[0]})_{ik}} - \sigma_{ki}^{[0]} = 0; \quad \frac{\partial \Phi}{\partial \boldsymbol{g}} = \boldsymbol{0}; \\
\frac{\partial \Phi}{\partial \theta} + \eta = 0; \quad q_i g_i \leq 0.
\end{aligned}
\tag{2.10}
$$

Remarks. From (2.10), we conclude the following

(1) $\frac{\partial \Phi}{\partial \boldsymbol{g}} = \boldsymbol{0}$ implies that Φ is not a function of $\boldsymbol{g}$.

(2) $\eta = -\frac{\partial \Phi}{\partial \theta}$ implies that η can be derived from Φ if Φ is known as a function of temperature, hence η cannot be a dependent variable in the constitutive theory.

(3) $\sigma_{ki}^{[0]} = \rho_0 \frac{\partial \Phi}{\partial (\varepsilon_{[0]})_{ik}}$ implies that $\boldsymbol{\sigma}^{[0]}$ can be determined from this relationship if Φ is known as a function of $\boldsymbol{\varepsilon}_{[0]}$.

(4) since Φ is not a function of $\boldsymbol{g}$, it implies that $\boldsymbol{\sigma}^{[0]}$ and η do not depend on $\boldsymbol{g}$ either.

Based on these remarks, we can conclude that Φ, $\boldsymbol{\sigma}^{[0]}$, and $\boldsymbol{q}$ are the only dependent variables in the constitutive theories for thermoelastic solid matter, and their argument tensors are as follows:

$$
\begin{aligned}
\Phi &= \Phi\big(\boldsymbol{\varepsilon}_{[0]}(x_i,t), \theta(x_i,t)\big) \\
\boldsymbol{\sigma}^{[0]} &= \boldsymbol{\sigma}^{[0]}\big(\boldsymbol{\varepsilon}_{[0]}(x_i,t), \theta(x_i,t)\big) \\
\boldsymbol{q} &= \boldsymbol{q}\big(\boldsymbol{\varepsilon}_{[0]}(x_i,t), \theta(x_i,t), \boldsymbol{g}(x_i,t)\big)
\end{aligned}
\tag{2.11}
$$

In (2.11) at this stage, $\boldsymbol{\varepsilon}_{[0]}$ is an argument tensor of $\boldsymbol{q}$, but the dependence of $\boldsymbol{q}$ on $\boldsymbol{\varepsilon}_{[0]}$ can be eliminated if so warranted by other considerations. Thus (2.11) and the first and last equations in (2.10) form the final set of equations that enable us to derive the constitutive theories for $\boldsymbol{\sigma}^{[0]}$ and $\boldsymbol{q}$.

2.3 Constitutive theories for stress tensor $\boldsymbol{\sigma}^{[0]}$

In this section, we consider various approaches of deriving constitutive theories for $\boldsymbol{\sigma}^{[0]}$ using (2.11) and the first equation in (2.10), *i.e.*

$$\Phi = \Phi(\boldsymbol{\varepsilon}_{[0]}, \theta) \tag{2.12}$$

$$\boldsymbol{\sigma}^{[0]} = \boldsymbol{\sigma}^{[0]}(\boldsymbol{\varepsilon}_{[0]}, \theta) \tag{2.13}$$

$$\sigma_{ki}^{[0]} = \rho_0 \frac{\partial \Phi}{\partial (\varepsilon_{[0]})_{ik}} \tag{2.14}$$

We consider three different approaches of deriving constitutive theories for $\boldsymbol{\sigma}^{[0]}$ (as described in the abstract and introduction).

2.3.1 Constitutive theory for $\sigma^{[0]}$ using (2.14) and assuming Φ to be a function of the invariants of $\boldsymbol{\varepsilon}_{[0]}$

In this approach, we consider Φ to be a function of the invariants $I_{\boldsymbol{\varepsilon}_{[0]}}$, $II_{\boldsymbol{\varepsilon}_{[0]}}$, and $III_{\boldsymbol{\varepsilon}_{[0]}}$ of $\boldsymbol{\varepsilon}_{[0]}$ and temperature θ [1–3] in the current configuration and then use (2.14) to determine the constitutive theory for the stress tensor $\boldsymbol{\sigma}^{[0]}$.

$$\Phi = \Phi(I_{\boldsymbol{\varepsilon}_{[0]}}, II_{\boldsymbol{\varepsilon}_{[0]}}, III_{\boldsymbol{\varepsilon}_{[0]}}, \theta) \tag{2.15}$$

in which

$$\begin{aligned} I_{\boldsymbol{\varepsilon}_{[0]}} &= (\varepsilon_{[0]})_{ii} = \mathrm{tr}(\boldsymbol{\varepsilon}_{[0]}) \\ II_{\boldsymbol{\varepsilon}_{[0]}} &= \frac{1}{2}(-(\varepsilon_{[0]})_{kl}(\varepsilon_{[0]})_{lk} + (\varepsilon_{[0]})_{ll}(\varepsilon_{[0]})_{kk}) \\ III_{\boldsymbol{\varepsilon}_{[0]}} &= \det\,(\boldsymbol{\varepsilon}_{[0]}) \end{aligned} \tag{2.16}$$

Using (2.15) and (2.14)

$$\boldsymbol{\sigma}^{[0]} = \rho_0 \left(\frac{\partial \Phi}{\partial I_{\boldsymbol{\varepsilon}_{[0]}}} \frac{\partial I_{\boldsymbol{\varepsilon}_{[0]}}}{\partial \boldsymbol{\varepsilon}_{[0]}} + \frac{\partial \Phi}{\partial II_{\boldsymbol{\varepsilon}_{[0]}}} \frac{\partial II_{\boldsymbol{\varepsilon}_{[0]}}}{\partial \boldsymbol{\varepsilon}_{[0]}} + \frac{\partial \Phi}{\partial III_{\boldsymbol{\varepsilon}_{[0]}}} \frac{\partial III_{\boldsymbol{\varepsilon}_{[0]}}}{\partial \boldsymbol{\varepsilon}_{[0]}} \right) \tag{2.17}$$

Using (2.16), it is straightforward to show

$$\frac{\partial I_{\boldsymbol{\varepsilon}_{[0]}}}{\partial \boldsymbol{\varepsilon}_{[0]}} = \boldsymbol{I} \text{ or } \frac{\partial I_{\boldsymbol{\varepsilon}_{[0]}}}{\partial (\varepsilon_{[0]})_{ij}} = \delta_{ij} \tag{2.18}$$

$$\frac{\partial II_{\boldsymbol{\varepsilon}_{[0]}}}{\partial \boldsymbol{\varepsilon}_{[0]}} = -\boldsymbol{\varepsilon}_{[0]} + I_{\boldsymbol{\varepsilon}_{[0]}} \boldsymbol{I} \tag{2.19}$$

$$\frac{\partial III_{\boldsymbol{\varepsilon}_{[0]}}}{\partial \boldsymbol{\varepsilon}_{[0]}} = III_{\boldsymbol{\varepsilon}_{[0]}} [\boldsymbol{\varepsilon}_{[0]}]^{-1} \tag{2.20}$$

Substituting from (2.18) to (2.20) into (2.17)

$$\boldsymbol{\sigma}^{[0]} = \rho_0 \left(\frac{\partial \Phi}{\partial I_{\boldsymbol{\varepsilon}_{[0]}}} \boldsymbol{I} + \frac{\partial \Phi}{\partial II_{\boldsymbol{\varepsilon}_{[0]}}} \big(-\boldsymbol{\varepsilon}_{[0]} + I_{\boldsymbol{\varepsilon}_{[0]}} \boldsymbol{I} \big) + \frac{\partial \Phi}{\partial III_{\boldsymbol{\varepsilon}_{[0]}}} III_{\boldsymbol{\varepsilon}_{[0]}} [\boldsymbol{\varepsilon}_{[0]}]^{-1} \right) \tag{2.21}$$

Collecting coefficients of $\boldsymbol{I}$, $\boldsymbol{\varepsilon}_{[0]}$ and $[\boldsymbol{\varepsilon}_{[0]}]^{-1}$ in (2.21)

$$\boldsymbol{\sigma}^{[0]} = \rho_0 \left(\frac{\partial \Phi}{\partial I_{\boldsymbol{\varepsilon}_{[0]}}} + \frac{\partial \Phi}{\partial II_{\boldsymbol{\varepsilon}_{[0]}}} I_{\boldsymbol{\varepsilon}_{[0]}} \right) \boldsymbol{I} + \left(-\rho_0 \frac{\partial \Phi}{\partial II_{\boldsymbol{\varepsilon}_{[0]}}} \right) \boldsymbol{\varepsilon}_{[0]} + \left(\rho_0 \frac{\partial \Phi}{\partial III_{\boldsymbol{\varepsilon}_{[0]}}} \right) [\boldsymbol{\varepsilon}_{[0]}]^{-1} \tag{2.22}$$

Let

$$\begin{aligned} {}^{\sigma}\alpha^0 &= \rho_0 \left(\frac{\partial \Phi}{\partial I_{\boldsymbol{\varepsilon}_{[0]}}} + \frac{\partial \Phi}{\partial II_{\boldsymbol{\varepsilon}_{[0]}}} I_{\boldsymbol{\varepsilon}_{[0]}} \right) \\ {}^{\sigma}\alpha^1 &= \left(-\rho_0 \frac{\partial \Phi}{\partial II_{\boldsymbol{\varepsilon}_{[0]}}} \right) \\ {}^{\sigma}\alpha^2 &= \left(\rho_0 \frac{\partial \Phi}{\partial III_{\boldsymbol{\varepsilon}_{[0]}}} \right). \end{aligned} \tag{2.23}$$

Then

$$\boldsymbol{\sigma}^{[0]} = {}^{\sigma}\alpha^0 \boldsymbol{I} + {}^{\sigma}\alpha^1 \boldsymbol{\varepsilon}_{[0]} + {}^{\sigma}\alpha^2 [\boldsymbol{\varepsilon}_{[0]}]^{-1} \tag{2.24}$$

$[\boldsymbol{\varepsilon}_{[0]}]^{-1}$ in (2.24) can be substituted in terms of $\boldsymbol{I}$, $\boldsymbol{\varepsilon}_{[0]}$, and $(\boldsymbol{\varepsilon}_{[0]})^2$ using the Hamilton-Cayley theorem [1] to obtain

$$\boldsymbol{\sigma}^{[0]} = {}^{\sigma}\widetilde{\alpha}^0 \boldsymbol{I} + {}^{\sigma}\widetilde{\alpha}^1 \boldsymbol{\varepsilon}_{[0]} + {}^{\sigma}\widetilde{\alpha}^2 (\boldsymbol{\varepsilon}_{[0]})^2 \tag{2.25}$$

in which ${}^{\sigma}\widetilde{\alpha}^i;\ i=0,1,2$ are functions of ${}^{\sigma}\alpha^i;\ i=0,1,2$, the invariants $I_{\boldsymbol{\varepsilon}_{[0]}}$, $II_{\boldsymbol{\varepsilon}_{[0]}}$, and $III_{\boldsymbol{\varepsilon}_{[0]}}$, and the temperature θ. Thus, the coefficients ${}^{\sigma}\widetilde{\alpha}^i = {}^{\sigma}\widetilde{\alpha}^i(I_{\boldsymbol{\varepsilon}_{[0]}}, II_{\boldsymbol{\varepsilon}_{[0]}}, III_{\boldsymbol{\varepsilon}_{[0]}}, \theta);\ i=0,1,2$. We note that ρ_0 is in the reference configuration (hence fixed or constant), but $I_{\boldsymbol{\varepsilon}_{[0]}}$, $II_{\boldsymbol{\varepsilon}_{[0]}}$, $III_{\boldsymbol{\varepsilon}_{[0]}}$, and θ are in the current configuration. The constitutive theory (2.25) is not usable yet due to the fact that ${}^{\sigma}\widetilde{\alpha}^i; i=0,1,2$ are functions of unknown deformation in the current configuration, hence are not known. We postpone further details of determining material coefficients using (2.25) until a later section. However, (2.25) is a fundamental form of the constitutive theory for $\boldsymbol{\sigma}^{[0]}$ as a function of $\boldsymbol{\varepsilon}_{[0]}$.

2.3.2 Constitutive theory for $\sigma^{[0]}$ using (2.13) and theory of generators and invariants

Consider (2.13), *i.e.*

$$\boldsymbol{\sigma}^{[0]} = \boldsymbol{\sigma}^{[0]}(\boldsymbol{\varepsilon}_{[0]}, \theta) \tag{2.26}$$

$\boldsymbol{\sigma}^{[0]}$ is a symmetric tensor of rank two whose argument tensors are $\boldsymbol{\varepsilon}_{[0]}$, a symmetric tensor of rank two, and θ, a tensor of rank zero. Based on theory of generators and invariants [1–3], $\boldsymbol{\sigma}^{[0]}$ can be expressed as a linear combination of $\boldsymbol{I}$, and the combined generators of its arguments, which in this case are generators of $\boldsymbol{\varepsilon}_{[0]}$ that are symmetric tensors of rank two. Between the argument tensors $\boldsymbol{\varepsilon}_{[0]}$ and θ, the combined generators that are symmetric tensors of rank two are $\boldsymbol{\varepsilon}_{[0]}$ and $(\boldsymbol{\varepsilon}_{[0]})^2$. A complete list of generators and invariants can be found in Appendix A. Using the same coefficients in the linear combination as appear in (2.25), we can write

$$\boldsymbol{\sigma}^{[0]} = {}^{\sigma}\widetilde{\alpha}^0 \boldsymbol{I} + {}^{\sigma}\widetilde{\alpha}^1 \boldsymbol{\varepsilon}_{[0]} + {}^{\sigma}\widetilde{\alpha}^2 (\boldsymbol{\varepsilon}_{[0]})^2 \tag{2.27}$$

in which the coefficients ${}^{\sigma}\widetilde{\alpha}^i;\ i=0,1,2$ are functions of $I_{\boldsymbol{\varepsilon}_{[0]}}$, $II_{\boldsymbol{\varepsilon}_{[0]}}$, $III_{\boldsymbol{\varepsilon}_{[0]}}$, and θ in the current configuration, *i.e.*

$${}^{\sigma}\widetilde{\alpha}^i = {}^{\sigma}\widetilde{\alpha}^i(I_{\boldsymbol{\varepsilon}_{[0]}}, II_{\boldsymbol{\varepsilon}_{[0]}}, III_{\boldsymbol{\varepsilon}_{[0]}}, \theta);\ i=0,1,2 \tag{2.28}$$

We note that (2.27) is the same as (2.25) from the first approach in section 2.3.1 with the same definition of the coefficients. Thus, the remarks made in section 2.3.1 regarding the coefficients hold here as well. When using the theory of generators and invariants, it is easier to use the principal invariants $i_{\varepsilon_{[0]}}$, $ii_{\varepsilon_{[0]}}$, and $iii_{\varepsilon_{[0]}}$ instead of $I_{\varepsilon_{[0]}}$, $II_{\varepsilon_{[0]}}$, and $III_{\varepsilon_{[0]}}$ in (2.28). Since the two sets of invariants are related [1], the final outcome remains the same as in section 2.3.1.

2.3.3 Constitutive theory for $\sigma^{[0]}$ by expanding Φ in Taylor series in $\varepsilon_{[0]}$ about a known configuration $\underline{\Omega}$ and then using (2.14)

We consider $\Phi = \Phi(\varepsilon_{[0]}, \theta)$ and expand Φ in $\varepsilon_{[0]}$ using Taylor series about a known configuration $\underline{\Omega}$.

$$
\begin{aligned}
\Phi = \Phi|_{\underline{\Omega}} &+ \left.\frac{\partial \Phi}{\partial (\varepsilon_{[0]})_{ij}}\right|_{\underline{\Omega}} \left((\varepsilon_{[0]})_{ij} - ((\varepsilon_{[0]})_{ij})_{\underline{\Omega}}\right) \\
&+ \frac{1}{2!}\left.\frac{\partial^2 \Phi}{\partial (\varepsilon_{[0]})_{ij}\partial (\varepsilon_{[0]})_{kl}}\right|_{\underline{\Omega}} \left((\varepsilon_{[0]})_{ij} - ((\varepsilon_{[0]})_{ij})_{\underline{\Omega}}\right) \\
&\qquad \left((\varepsilon_{[0]})_{kl} - ((\varepsilon_{[0]})_{kl})_{\underline{\Omega}}\right) \\
&+ \frac{1}{3!}\left.\frac{\partial^3 \Phi}{\partial (\varepsilon_{[0]})_{ij}\partial (\varepsilon_{[0]})_{kl}\partial (\varepsilon_{[0]})_{pq}}\right|_{\underline{\Omega}} \left((\varepsilon_{[0]})_{ij} - ((\varepsilon_{[0]})_{ij})_{\underline{\Omega}}\right) \\
&\qquad \left((\varepsilon_{[0]})_{kl} - ((\varepsilon_{[0]})_{kl})_{\underline{\Omega}}\right)\left((\varepsilon_{[0]})_{pq} - ((\varepsilon_{[0]})_{pq})_{\underline{\Omega}}\right) \\
&\qquad + \cdots
\end{aligned} \tag{2.29}
$$

Let

$$
\begin{aligned}
&\Phi|_{\underline{\Omega}} = C; \qquad \left.\frac{\partial \Phi}{\partial (\varepsilon_{[0]})_{ij}}\right|_{\underline{\Omega}} = C_{ij}; \\
&\left.\frac{\partial \Phi}{\partial (\varepsilon_{[0]})_{ij}\partial (\varepsilon_{[0]})_{kl}}\right|_{\underline{\Omega}} = \hat{C}_{ijkl}; \quad \left.\frac{\partial \Phi}{\partial (\varepsilon_{[0]})_{ij}\partial (\varepsilon_{[0]})_{kl}\partial (\varepsilon_{[0]})_{pq}}\right|_{\underline{\Omega}} = \widetilde{C}_{ijklpq}
\end{aligned} \tag{2.30}
$$

The coefficients in (2.30) are defined in the known configuration $\underline{\Omega}$. Substituting from

(2.30) into (2.29):

$$\Phi = C + C_{ij}\big((\varepsilon_{[0]})_{ij} - ((\varepsilon_{[0]})_{ij})_{\underline{\Omega}}\big) + \hat{C}_{ijkl}\big((\varepsilon_{[0]})_{ij} - ((\varepsilon_{[0]})_{ij})_{\underline{\Omega}}\big)\big((\varepsilon_{[0]})_{kl} - ((\varepsilon_{[0]})_{kl})_{\underline{\Omega}}\big)$$
$$+ \frac{1}{3!}\widetilde{C}_{ijklpq}\big((\varepsilon_{[0]})_{ij} - ((\varepsilon_{[0]})_{ij})_{\underline{\Omega}}\big)\big((\varepsilon_{[0]})_{kl} - ((\varepsilon_{[0]})_{kl})_{\underline{\Omega}}\big)\big((\varepsilon_{[0]})_{pq} - ((\varepsilon_{[0]})_{pq})_{\underline{\Omega}}\big) + \cdots \quad (2.31)$$

Differentiating Φ with respect to $\boldsymbol{\varepsilon}_{[0]}$ and using (2.14) and noting that partial derivatives of the coefficients in (2.31) are zero and

$$\frac{\partial}{\partial(\varepsilon_{[0]})_{mn}}\big((\varepsilon_{[0]})_{ij} - ((\varepsilon_{[0]})_{ij})_{\underline{\Omega}}\big) = \delta_{im}\delta_{jn}$$

$$\begin{aligned}\frac{\partial}{\partial(\varepsilon_{[0]})_{mn}}&\big((\varepsilon_{[0]})_{ij} - ((\varepsilon_{[0]})_{ij})_{\underline{\Omega}}\big)\big((\varepsilon_{[0]})_{kl} - ((\varepsilon_{[0]})_{kl})_{\underline{\Omega}}\big)\\ &= \delta_{im}\delta_{jn}\big((\varepsilon_{[0]})_{kl} - ((\varepsilon_{[0]})_{kl})_{\underline{\Omega}}\big)\\ &\quad + \delta_{km}\delta_{ln}\big((\varepsilon_{[0]})_{ij} - ((\varepsilon_{[0]})_{ij})_{\underline{\Omega}}\big)\end{aligned} \quad (2.32)$$

$$\begin{aligned}\frac{\partial}{\partial(\varepsilon_{[0]})_{mn}}&\big((\varepsilon_{[0]})_{ij} - ((\varepsilon_{[0]})_{ij})_{\underline{\Omega}}\big)\big((\varepsilon_{[0]})_{kl} - ((\varepsilon_{[0]})_{kl})_{\underline{\Omega}}\big)\big((\varepsilon_{[0]})_{pq} - ((\varepsilon_{[0]})_{pq})_{\underline{\Omega}}\big)\\ &= \delta_{im}\delta_{jn}\big((\varepsilon_{[0]})_{kl} - ((\varepsilon_{[0]})_{kl})_{\underline{\Omega}}\big)\big((\varepsilon_{[0]})_{pq} - ((\varepsilon_{[0]})_{pq})_{\underline{\Omega}}\big)\\ &\quad + \delta_{km}\delta_{ln}\big((\varepsilon_{[0]})_{ij} - ((\varepsilon_{[0]})_{ij})_{\underline{\Omega}}\big)\big((\varepsilon_{[0]})_{pq} - ((\varepsilon_{[0]})_{pq})_{\underline{\Omega}}\big)\\ &\quad + \delta_{pm}\delta_{qn}\big((\varepsilon_{[0]})_{ij} - ((\varepsilon_{[0]})_{ij})_{\underline{\Omega}}\big)\big((\varepsilon_{[0]})_{kl} - ((\varepsilon_{[0]})_{kl})_{\underline{\Omega}}\big)\end{aligned}$$

We obtain (2.33) if we substitute from (2.32) into (2.14). In doing so, (i) we collect all terms in configuration $\underline{\Omega}$; (ii) we define coefficients of $\boldsymbol{\varepsilon}_{[0]}$ and $(\boldsymbol{\varepsilon}_{[0]})^2$ (those that are defined in the known configuration $\underline{\Omega}$; and (iii) we use symmetry of the coefficients, *i.e.*, $\hat{C}_{mnij} = \hat{C}_{ijmn}$, etc.

$$\sigma^{[0]}_{mn} = (\sigma^{[0]}_{mn})_{\underline{\Omega}} + \underline{\underline{C}}_{mnij}(\varepsilon_{[0]})_{ij} + \overline{\overline{C}}_{mnijkl}(\varepsilon_{[0]})_{ij}(\varepsilon_{[0]})_{kl} + \cdots \quad (2.33)$$

Remarks. (1) We note that (2.25), (2.27), and (2.33) are similar in the sense that all these three forms of the constitutive theory for $\boldsymbol{\sigma}^{[0]}$ contain exactly the same tensors on both sides of the equality that are defined in the current configuration.

(2) In (2.25) and (2.27), the coefficients ${}^{\sigma}\alpha^i;\ i = 0,1,2$ are in the current configuration, whereas in (2.33) are in the known configuration $\underline{\Omega}$.

(3) Based on the derivations given in sections 2.3.1 and 2.3.2, it is clear that the Taylor series expansion in (2.29) must be limited up to the cubic terms in $\boldsymbol{\varepsilon}_{[0]}$. Inclusion of further higher degree terms in $\boldsymbol{\varepsilon}_{[0]}$ is non-physical as it is not supported by the derivations in sections 2.3.1 and 2.3.2 that are strictly based on entropy inequality.

(4) From Taylor series expansion, it is clear that the coefficients in (2.33) are functions of $\boldsymbol{\varepsilon}_{[0]}$ and θ in a known configuration $\underline{\Omega}$, whereas the coefficients ${}^{\sigma}\widetilde{\alpha}^i;\ i = 0,1,2$ in (2.25) and (2.27) are functions of the invariants of $\boldsymbol{\varepsilon}_{[0]}$ and temperature θ in the current configuration. The coefficients in (2.33) are in fact the material coefficients, whereas in (2.25) and (2.27), the material coefficients are yet to be defined using ${}^{\sigma}\widetilde{\alpha}^i;\ i = 0,1,2$.

(5) The issue of whether (2.33) is superior over (2.25) or (2.27) and vice versa can only be addressed after we determine the material coefficients in (2.25) or (2.27) using ${}^{\sigma}\widetilde{\alpha}^i;\ i = 0,1,2$. We present details in the following section.

(6) For homogeneous and isotropic solid matter, the coefficients in (2.33) can be simplified [1].

2.3.4 Definition of material coefficients using ${}^{\sigma}\widetilde{\alpha}^i;\ i = 0,1,2$ in (2.25) or (2.27)

Consider

$$\boldsymbol{\sigma}^{[0]} = {}^{\sigma}\widetilde{\alpha}^0 \boldsymbol{I} + {}^{\sigma}\widetilde{\alpha}^1 \boldsymbol{\varepsilon}_{[0]} + {}^{\sigma}\widetilde{\alpha}^2 (\boldsymbol{\varepsilon}_{[0]})^2. \tag{2.34}$$

We consider ${}^{\sigma}\widetilde{\alpha}^{i};\ i = 0,1,2$ to be functions of $I_{\boldsymbol{\varepsilon}_{[0]}}$, $II_{\boldsymbol{\varepsilon}_{[0]}}$, $III_{\boldsymbol{\varepsilon}_{[0]}}$ (as opposed to principal invariants) and temperature θ

$$
{}^{\sigma}\widetilde{\alpha}^{i} = {}^{\sigma}\widetilde{\alpha}^{i}(I_{\boldsymbol{\varepsilon}_{[0]}}, II_{\boldsymbol{\varepsilon}_{[0]}}, III_{\boldsymbol{\varepsilon}_{[0]}}, \theta);\ i = 0,1,2 \tag{2.35}
$$

We can expand ${}^{\sigma}\widetilde{\alpha}^{i}$ in Taylor series in $I_{\boldsymbol{\varepsilon}_{[0]}}$, $II_{\boldsymbol{\varepsilon}_{[0]}}$, $III_{\boldsymbol{\varepsilon}_{[0]}}$, and θ about a known configuration $\underline{\Omega}$. We retain only up to linear terms in the invariants of $\boldsymbol{\varepsilon}_{[0]}$ and temperature θ in the Taylor series expansion. We introduce the following notation to make the presentation compact.

$$
{}^{\sigma}\underset{\sim}{I}^{1} = I_{\boldsymbol{\varepsilon}_{[0]}};\quad {}^{\sigma}\underset{\sim}{I}^{2} = II_{\boldsymbol{\varepsilon}_{[0]}};\quad {}^{\sigma}\underset{\sim}{I}^{3} = III_{\boldsymbol{\varepsilon}_{[0]}}. \tag{2.36}
$$

Using the notation in (2.36), we can write

$$
{}^{\sigma}\widetilde{\alpha}^{i} = {}^{\sigma}\widetilde{\alpha}^{i}|_{\underline{\Omega}} + \sum_{j=1}^{3} \left.\frac{\partial\, {}^{\sigma}\widetilde{\alpha}^{i}}{\partial\, {}^{\sigma}\underset{\sim}{I}^{j}}\right|_{\underline{\Omega}} \left({}^{\sigma}\underset{\sim}{I}^{j} - ({}^{\sigma}\underset{\sim}{I}^{j})_{\underline{\Omega}}\right) + \left.\frac{\partial\, {}^{\sigma}\widetilde{\alpha}^{i}}{\partial\theta}\right|_{\underline{\Omega}} (\theta - \theta_{\underline{\Omega}});\ i = 0,1,2 \tag{2.37}
$$

Substituting from (2.37) into (2.34)

$$
\begin{aligned}
\boldsymbol{\sigma}^{[0]} = &\left({}^{\sigma}\widetilde{\alpha}^{0}|_{\underline{\Omega}} + \sum_{j=1}^{3} \left.\frac{\partial {}^{\sigma}\widetilde{\alpha}^{0}}{\partial {}^{\sigma}\underset{\sim}{I}^{j}}\right|_{\underline{\Omega}} \left({}^{\sigma}\underset{\sim}{I}^{j} - ({}^{\sigma}\underset{\sim}{I}^{j})_{\underline{\Omega}}\right) + \left.\frac{\partial {}^{\sigma}\widetilde{\alpha}^{0}}{\partial\theta}\right|_{\underline{\Omega}} (\theta - \theta_{\underline{\Omega}})\right) \boldsymbol{I} + \\
&\left({}^{\sigma}\widetilde{\alpha}^{1}|_{\underline{\Omega}} + \sum_{j=1}^{3} \left.\frac{\partial {}^{\sigma}\widetilde{\alpha}^{1}}{\partial {}^{\sigma}\underset{\sim}{I}^{j}}\right|_{\underline{\Omega}} \left({}^{\sigma}\underset{\sim}{I}^{j} - ({}^{\sigma}\underset{\sim}{I}^{j})_{\underline{\Omega}}\right) + \left.\frac{\partial {}^{\sigma}\widetilde{\alpha}^{1}}{\partial\theta}\right|_{\underline{\Omega}} (\theta - \theta_{\underline{\Omega}})\right) \boldsymbol{\varepsilon}_{[0]} + \\
&\left({}^{\sigma}\widetilde{\alpha}^{2}|_{\underline{\Omega}} + \sum_{j=1}^{3} \left.\frac{\partial {}^{\sigma}\widetilde{\alpha}^{2}}{\partial {}^{\sigma}\underset{\sim}{I}^{j}}\right|_{\underline{\Omega}} \left({}^{\sigma}\underset{\sim}{I}^{j} - ({}^{\sigma}\underset{\sim}{I}^{j})_{\underline{\Omega}}\right) + \left.\frac{\partial {}^{\sigma}\widetilde{\alpha}^{2}}{\partial\theta}\right|_{\underline{\Omega}} (\theta - \theta_{\underline{\Omega}})\right) (\boldsymbol{\varepsilon}_{[0]})^{2}
\end{aligned} \tag{2.38}
$$

Collecting coefficients defined in configuration $\underline{\Omega}$ of $\boldsymbol{I}$, $\boldsymbol{\varepsilon}_{[0]}$, ${}^{\sigma}\underset{\sim}{I}^{j}\boldsymbol{I};\ j = 1,2,3$, ${}^{\sigma}\underset{\sim}{I}^{j}\boldsymbol{\varepsilon}_{[0]};\ j = 1,2,3$, ${}^{\sigma}\underset{\sim}{I}^{j}(\boldsymbol{\varepsilon}_{[0]})^{2};\ j = 1,2,3$, $(\theta - \theta_{\underline{\Omega}})\boldsymbol{I}$, $(\theta - \theta_{\underline{\Omega}})\boldsymbol{\varepsilon}_{[0]}$, and $(\theta - \theta_{\underline{\Omega}})(\boldsymbol{\varepsilon}_{[0]})^{2}$, we can write the following using (2.38)

$$
\begin{aligned}
\boldsymbol{\sigma}^{[0]} = & \left({}^{\sigma}\widetilde{\alpha}^{0}|_{\underline{\Omega}} - \sum_{j=1}^{3} \frac{\partial\, {}^{\sigma}\widetilde{\alpha}^{0}}{\partial\, {}^{\sigma}\underset{\sim}{I}^{j}} \bigg| ({}^{\sigma}\underset{\sim}{I}^{j})_{\underline{\Omega}} \right) \boldsymbol{I} + \left({}^{\sigma}\widetilde{\alpha}^{1}|_{\underline{\Omega}} - \sum_{j=1}^{3} \frac{\partial\, {}^{\sigma}\widetilde{\alpha}^{1}}{\partial\, {}^{\sigma}\underset{\sim}{I}^{j}} \bigg| ({}^{\sigma}\underset{\sim}{I}^{j})_{\underline{\Omega}} \right) \boldsymbol{\varepsilon}_{[0]} \\
& + \left({}^{\sigma}\widetilde{\alpha}^{2}|_{\underline{\Omega}} - \sum_{j=1}^{3} \frac{\partial\, {}^{\sigma}\widetilde{\alpha}^{2}}{\partial\, {}^{\sigma}\underset{\sim}{I}^{j}} \bigg| ({}^{\sigma}\underset{\sim}{I}^{j})_{\underline{\Omega}} \right) (\boldsymbol{\varepsilon}_{[0]})^2 + \sum_{j=1}^{3} \frac{\partial\, {}^{\sigma}\widetilde{\alpha}^{0}}{\partial\, {}^{\sigma}\underset{\sim}{I}^{j}} \bigg|_{\underline{\Omega}} \left({}^{\sigma}\underset{\sim}{I}^{j} \boldsymbol{I} \right) \\
& + \sum_{j=1}^{3} \frac{\partial\, {}^{\sigma}\widetilde{\alpha}^{1}}{\partial\, {}^{\sigma}\underset{\sim}{I}^{j}} \bigg|_{\underline{\Omega}} \left({}^{\sigma}\underset{\sim}{I}^{j} \boldsymbol{\varepsilon}_{[0]} \right) + \sum_{j=1}^{3} \frac{\partial\, {}^{\sigma}\widetilde{\alpha}^{2}}{\partial\, {}^{\sigma}\underset{\sim}{I}^{j}} \bigg|_{\underline{\Omega}} \left({}^{\sigma}\underset{\sim}{I}^{j} (\boldsymbol{\varepsilon}_{[0]})^2 \right) + \frac{\partial\, {}^{\sigma}\widetilde{\alpha}^{0}}{\partial \theta} \bigg|_{\underline{\Omega}} \left((\theta - \theta_{\underline{\Omega}}) \boldsymbol{I} \right) \\
& + \frac{\partial\, {}^{\sigma}\widetilde{\alpha}^{1}}{\partial \theta} \bigg|_{\underline{\Omega}} \left((\theta - \theta_{\underline{\Omega}}) \boldsymbol{\varepsilon}_{[0]} \right) + \frac{\partial\, {}^{\sigma}\widetilde{\alpha}^{2}}{\partial \theta} \bigg|_{\underline{\Omega}} \left((\theta - \theta_{\underline{\Omega}}) (\boldsymbol{\varepsilon}_{[0]})^2 \right)
\end{aligned}
\tag{2.39}
$$

Let us define

$$
\begin{aligned}
& \sigma_0|_{\underline{\Omega}} = {}^{\sigma}\widetilde{\alpha}^{0}|_{\underline{\Omega}} - \sum_{j=1}^{3} \frac{\partial\, {}^{\sigma}\widetilde{\alpha}^{0}}{\partial\, {}^{\sigma}\underset{\sim}{I}^{j}} \bigg|_{\underline{\Omega}} ({}^{\sigma}\underset{\sim}{I}^{j})_{\underline{\Omega}} && {}^{\sigma}\widetilde{a}_j = \frac{\partial\, {}^{\sigma}\widetilde{\alpha}^{0}}{\partial\, {}^{\sigma}\underset{\sim}{I}^{j}} \bigg|_{\underline{\Omega}} ;\ j = 1,2,3 \\
& {}^{\sigma}\widetilde{b}_i = {}^{\sigma}\widetilde{\alpha}^{i}|_{\underline{\Omega}} - \sum_{j=1}^{3} \frac{\partial\, {}^{\sigma}\widetilde{\alpha}^{i}}{\partial\, {}^{\sigma}\underset{\sim}{I}^{j}} \bigg|_{\underline{\Omega}} ;\ i = 1,2 && {}^{\sigma}\widetilde{c}_{1j} = \frac{\partial\, {}^{\sigma}\widetilde{\alpha}^{1}}{\partial\, {}^{\sigma}\underset{\sim}{I}^{j}} \bigg|_{\underline{\Omega}} ;\ j = 1,2,3 \\
& {}^{\sigma}\widetilde{c}_{2j} = \frac{\partial\, {}^{\sigma}\widetilde{\alpha}^{2}}{\partial\, {}^{\sigma}\underset{\sim}{I}^{j}} \bigg|_{\underline{\Omega}} ;\ j = 1,2,3 && {}^{\sigma}\widetilde{d}_1 = \frac{\partial\, {}^{\sigma}\widetilde{\alpha}^{1}}{\partial \theta} \bigg|_{\underline{\Omega}} \\
& {}^{\sigma}\widetilde{d}_2 = \frac{\partial\, {}^{\sigma}\widetilde{\alpha}^{2}}{\partial \theta} \bigg|_{\underline{\Omega}} && (\alpha_{tm})_{\underline{\Omega}} = -\frac{\partial\, {}^{\sigma}\widetilde{\alpha}^{0}}{\partial \theta} \bigg|_{\underline{\Omega}}
\end{aligned}
\tag{2.40}
$$

Substituting from (2.40) into (2.39)

$$
\begin{aligned}
\boldsymbol{\sigma}^{[0]} = & \sigma_0|_{\underline{\Omega}} \boldsymbol{I} + {}^{\sigma}\widetilde{b}_1 \boldsymbol{\varepsilon}_{[0]} + {}^{\sigma}\widetilde{b}_2 (\boldsymbol{\varepsilon}_{[0]})^2 + \\
& \sum_{j=1}^{3} {}^{\sigma}\widetilde{a}_j \left({}^{\sigma}\underset{\sim}{I}^{j} [i] \right) + \sum_{j=1}^{3} {}^{\sigma}\widetilde{c}_{1j} \left({}^{\sigma}\underset{\sim}{I}^{j} \boldsymbol{\varepsilon}_{[0]} \right) + \\
& \sum_{j=1}^{3} {}^{\sigma}\widetilde{c}_{2j} \left({}^{\sigma}\underset{\sim}{I}^{j} (\boldsymbol{\varepsilon}_{[0]})^2 \right) + {}^{\sigma}\widetilde{d}_1 (\theta - \theta_{\underline{\Omega}}) \boldsymbol{\varepsilon}_{[0]} + \\
& {}^{\sigma}\widetilde{d}_2 (\theta - \theta_{\underline{\Omega}}) (\boldsymbol{\varepsilon}_{[0]})^2 + (\alpha_{tm})_{\underline{\Omega}} (\theta - \theta_{\underline{\Omega}}) \boldsymbol{I}
\end{aligned}
\tag{2.41}
$$

$\sigma_0|_{\underline{\Omega}}$ is the initial stress in the configuration $\underline{\Omega}$. This constitutive theory requires determination of 14 material coefficients defined in (2.40) (all but excluding $\sigma_0|_{\underline{\Omega}}$), all evaluated in a known configuration $\underline{\Omega}$. The constitutive theory (2.41) for $\boldsymbol{\sigma}^{[0]}$ is the most general form of the constitutive theory for $\boldsymbol{\sigma}^{[0]}$ as a function of $\boldsymbol{\varepsilon}_{[0]}$ and temperature θ resulting from the entropy inequality or the theory of generators and invariants. This theory is based on integrity and hence complete, but it contains too many material coefficients to be determined, experimentally or otherwise.

Further Simplifications

In this section, we consider simplifications of the constitutive theory for $\boldsymbol{\sigma}^{[0]}$ given by (2.41). If we only consider a constitutive theory for $\boldsymbol{\sigma}^{[0]}$ that is linear in the components of $\boldsymbol{\varepsilon}_{[0]}$ and further neglect the $(\theta - \theta_{\underline{\Omega}})\boldsymbol{\varepsilon}_{[0]}$ terms, then (2.41) reduces to

$$\begin{aligned} \boldsymbol{\sigma}^{[0]} = & \sigma_0|_{\underline{\Omega}}\boldsymbol{I} + {}^{\sigma}\widetilde{b}_1\boldsymbol{\varepsilon}_{[0]} + {}^{\sigma}\widetilde{a}_1 \operatorname{tr}(\boldsymbol{\varepsilon}_{[0]})\boldsymbol{I} + \\ & (\alpha_{tm})_{\underline{\Omega}}(\theta - \theta_{\underline{\Omega}})\boldsymbol{I} \end{aligned} \tag{2.42}$$

This constitutive theory only requires three material coefficients, ${}^{\sigma}\widetilde{a}_1$, ${}^{\sigma}\widetilde{b}_1$, and α_{tm} in a known configuration $\underline{\Omega}$.

Remarks. (1) It is perhaps meaningful to compare the constitutive theory (2.33) resulting from the Taylor series expansion of Φ and the constitutive theory (2.41) from the entropy inequality or the theory of generators and invariants as the material coefficients in the two are now defined (in a known configuration).

(2) We observe that not all terms containing $\boldsymbol{\varepsilon}_{[0]}$ in the current configuration on the right hand side of (2.33) and (2.41) are the same.

(3) Furthermore, the coefficients in (2.33) are functions of $\boldsymbol{\varepsilon}_{[0]}|_{\underline{\Omega}}$ and $\theta|_{\underline{\Omega}}$ whereas the material coefficients in (2.41) are functions of the *invariants* of $\boldsymbol{\varepsilon}_{[0]}$ and temperature θ in $\underline{\Omega}$, hence in general, the two sets of material coefficients are different.

(4) Based on remarks 2 and 3, it is straightforward to conclude that the constitutive theories (2.33) and (2.41) are different. This raises the question of superiority of one over the other. The constitutive theories in sections 2.3.1 and 2.3.2 are strictly based on entropy inequality and integrity and hence are in precise agreement with the axioms and principles of continuum mechanics. The Taylor series expansion, though based on the principle of smooth neighborhood, ignores the fundamental axiom that the coefficients in the constitutive theories must be functions of the combined invariants of the argument tensors.

(5) Based on remark 4, the constitutive theories (2.33) and (2.41) are not equivalent and the constitutive theory (2.41) is meritorious over (2.33).

2.4 Constitutive theory for heat vector $\boldsymbol{q}$

The conditions resulting from the entropy inequality require that

$$q_i g_i \leq 0 \tag{2.43}$$

be satisfied by the constitutive theories for $\boldsymbol{q}$ regardless of how they are derived. We can take two approaches to derive constitutive theory for $\boldsymbol{q}$. In the first approach [1–3], we strictly use (2.43) to derive the constitutive theory for $\boldsymbol{q}$. Such constitutive theory for $\boldsymbol{q}$ will naturally satisfy the entropy inequality as it is derived using the conditions resulting from it. In the second approach we use the argument tensors of $\boldsymbol{q}$ and then use theory of generators and invariants. The constitutive theories derived using this approach must ensure that the constitutive theories for $\boldsymbol{q}$ satisfy (2.43) so that the deforming matter will be in thermodynamic equilibrium during evolution. We present the derivation of the constitutive theories for $\boldsymbol{q}$ using both approaches and present comparisons of the resulting constitutive theories, discuss assumptions, and make some remarks regarding their merits and shortcomings.

2.4.1 Constitutive theory for q using entropy inequality [1–3]

This derivation based on (2.43) is fundamental and can be found in any textbook on continuum mechanics. We present details in the following to point out the assumptions used in the derivation as they play a significant role when comparing this constitutive theory with the theories resulting from the theory of generators and invariants. Following references [1–3], we begin with (2.43). Equation (2.43) implies that

$$\boldsymbol{q}^T \boldsymbol{g} = \beta \leq 0 \tag{2.44}$$

Using equality, we obtain

$$\frac{\partial \beta}{\partial \boldsymbol{g}} = \boldsymbol{q} \text{ or } \frac{\partial \beta}{\partial g_i} = q_i \tag{2.45}$$

Hence

$$\boldsymbol{q}|_{\boldsymbol{g}=0} = \left.\frac{\partial \beta}{\partial \boldsymbol{g}}\right|_{\boldsymbol{g}=0} = 0 \tag{2.46}$$

That is, heat flux vanishes in the absence of temperature gradient. Thus, the constitutive theory for $\boldsymbol{q}$ must be a function of $\boldsymbol{g}$. At this stage, many possibilities exist; the simplest of course is assuming that $\boldsymbol{q}$ is proportional to $-\boldsymbol{g}$, *i.e.* $\boldsymbol{q}$ is a linear function of $-\boldsymbol{g}$.

$$q_i(\boldsymbol{g}) = -k_{ij}(\theta) g_j \text{ or } \boldsymbol{q} = -\big[k(\theta)\big]\boldsymbol{g} \tag{2.47}$$

from which we define

$$\frac{\partial q_i}{\partial g_j} = -k_{ij}(\theta) \tag{2.48}$$

Also, from (2.45)

$$\frac{\partial^2 \beta}{\partial g_j \partial g_i} = \frac{\partial q_i}{\partial g_j} = -k_{ij}(\theta) \leq 0 \tag{2.49}$$

From (2.49), we conclude that $[k]$ is positive-semidefinite and all its eigenvalues are non-negative. Equation (2.47) is the Fourier heat conduction law. The thermal conductivity matrix $[k]$ does not have to be symmetric but is often assumed to be. In general, in this constitutive theory for $\boldsymbol{q}$, the coefficients of $[k]$ can be functions of temperature θ. This constitutive theory is based on the assumption that $\boldsymbol{q}$ is a linear function of $\boldsymbol{g}$.

2.4.2 Constitutive theories for q using theory of generators and invariants

In this approach, the heat vector $\boldsymbol{q}$, a tensor of rank one, is expressed as a linear combination of the combined generators (only tensors of rank one) of its argument tensors. The material coefficients in the linear combination are assumed to be functions of the combined invariants of the argument tensors and temperature θ. The material coefficients are derived by expanding each coefficient in the linear combination in Taylor series about a known configuration. In this approach it is obvious that the explicit form of the constitutive theory for $\boldsymbol{q}$ depends on the argument tensors of $\boldsymbol{q}$ and the terms retained in the Taylor series expansion of the coefficients in the linear combination. We present derivations of two constitutive theories for $\boldsymbol{q}$ using this approach.

2.4.2.1 *Approach I*

In this derivation, we assume that

$$\boldsymbol{q} = \boldsymbol{q}(\boldsymbol{g}, \theta) \tag{2.50}$$

$\boldsymbol{q}$ and $\boldsymbol{g}$ are tensors of rank one and θ is a tensor of rank zero. The only combined generator of rank one of the argument tensors $\boldsymbol{g}$ and θ is $\boldsymbol{g}$, hence based on the theory of generators and invariants [1–3], we can write

$$\boldsymbol{q} = -{}^{q}\alpha \boldsymbol{g} \tag{2.51}$$

The coefficient ${}^{q}\alpha$ is a function of the combined invariants of $\boldsymbol{g}$, θ, *i.e.*, $\boldsymbol{g} \cdot \boldsymbol{g}$ and temperature θ. Let us define ${}^{q}\underset{\sim}{I} = \boldsymbol{g} \cdot \boldsymbol{g}$ to simplify the details of further derivation. We note that (2.51) holds in the current configuration in which the deformation is not known. Hence in (2.51), ${}^{q}\alpha = {}^{q}\alpha({}^{q}\underset{\sim}{I}, \theta)$ is not yet deterministic. To determine material coefficients from (2.51), we expand ${}^{q}\alpha({}^{q}\underset{\sim}{I}, \theta)$ in Taylor series about a known configuration $\underline{\Omega}$ in ${}^{q}\underset{\sim}{I}$ and θ and retain only up to linear terms in ${}^{q}\underset{\sim}{\underline{I}}$ and θ.

$$
{}^{q}\alpha = {}^{q}\alpha|_{\underline{\Omega}} + \left.\frac{\partial\, {}^{q}\alpha}{\partial\, {}^{q}\underset{\sim}{\underline{I}}}\right|_{\underline{\Omega}} \left({}^{q}\underset{\sim}{\underline{I}} - ({}^{q}\underset{\sim}{\underline{I}})_{\underline{\Omega}}\right) + \left.\frac{\partial\, {}^{q}\alpha}{\partial\theta}\right|_{\underline{\Omega}} (\theta - \theta_{\underline{\Omega}}) \tag{2.52}
$$

Substituting from (2.52) into (2.51)

$$
\boldsymbol{q} = \left({}^{q}\alpha|_{\underline{\Omega}} + \left.\frac{\partial\, {}^{q}\alpha}{\partial\, {}^{q}\underset{\sim}{\underline{I}}}\right|_{\underline{\Omega}} \left({}^{q}\underset{\sim}{\underline{I}} - ({}^{q}\underset{\sim}{\underline{I}})_{\underline{\Omega}}\right) + \left.\frac{\partial\, {}^{q}\alpha}{\partial\theta}\right|_{\underline{\Omega}} (\theta - \theta_{\underline{\Omega}}) \right) \boldsymbol{g} \tag{2.53}
$$

We note that ${}^{q}\alpha|_{\underline{\Omega}}$, $\left.\frac{\partial\, {}^{q}\alpha}{\partial\, {}^{q}\underset{\sim}{\underline{I}}}\right|_{\underline{\Omega}}$, and $\left.\frac{\partial\, {}^{q}\alpha}{\partial\theta}\right|_{\underline{\Omega}}$ are functions of ${}^{q}\underset{\sim}{\underline{I}}|_{\underline{\Omega}}$ and $\theta|_{\underline{\Omega}}$, whereas ${}^{q}\alpha$ in (2.51) is a function of ${}^{q}\underset{\sim}{\underline{I}}$ and θ in the current configuration. From (2.53) we can write the following, noting that ${}^{q}\underset{\sim}{\underline{I}} = \boldsymbol{g}\cdot\boldsymbol{g}$

$$
\boldsymbol{q} = -{}^{q}\alpha|_{\underline{\Omega}}\,\boldsymbol{g} - \left(\left.\frac{\partial\, {}^{q}\alpha}{\partial\, {}^{q}\underset{\sim}{\underline{I}}}\right|_{\underline{\Omega}} \right) (\boldsymbol{g}\cdot\boldsymbol{g})\boldsymbol{g} + \left.\frac{\partial\, {}^{q}\alpha}{\partial\, {}^{q}\underset{\sim}{\underline{I}}}\right|_{\underline{\Omega}} (\boldsymbol{g}\cdot\boldsymbol{g})_{\underline{\Omega}}\,\boldsymbol{g} - \left.\frac{\partial\, {}^{q}\alpha}{\partial\theta}\right|_{\underline{\Omega}} (\theta - \theta_{\underline{\Omega}})\boldsymbol{g} \tag{2.54}
$$

or

$$
\boldsymbol{q} = \left({}^{q}\alpha|_{\underline{\Omega}} - \left.\frac{\partial\, {}^{q}\alpha}{\partial\, {}^{q}\underset{\sim}{\underline{I}}}\right|_{\underline{\Omega}} (\boldsymbol{g}\cdot\boldsymbol{g})_{\underline{\Omega}} \right) \boldsymbol{g} - \left(\left.\frac{\partial\, {}^{q}\alpha}{\partial\, {}^{q}\underset{\sim}{\underline{I}}}\right|_{\underline{\Omega}} \right) (\boldsymbol{g}\cdot\boldsymbol{g})\boldsymbol{g} - \left.\frac{\partial\, {}^{q}\alpha}{\partial\theta}\right|_{\underline{\Omega}} (\theta - \theta_{\underline{\Omega}})\boldsymbol{g} \tag{2.55}
$$

Let

$$
\begin{aligned}
k(\theta_{\underline{\Omega}}, {}^{q}\underset{\sim}{\underline{I}}|_{\underline{\Omega}}) &= {}^{q}\alpha|_{\underline{\Omega}} + \left.\frac{\partial\, {}^{q}\alpha}{\partial\, {}^{q}\underset{\sim}{\underline{I}}}\right|_{\underline{\Omega}} (\boldsymbol{g}\cdot\boldsymbol{g})_{\underline{\Omega}} \\
k_1(\theta_{\underline{\Omega}}, {}^{q}\underset{\sim}{\underline{I}}|_{\underline{\Omega}}) &= \left.\frac{\partial\, {}^{q}\alpha}{\partial\, {}^{q}\underset{\sim}{\underline{I}}}\right|_{\underline{\Omega}} \\
k_2(\theta_{\underline{\Omega}}, {}^{q}\underset{\sim}{\underline{I}}|_{\underline{\Omega}}) &= \left.\frac{\partial\, {}^{q}\alpha}{\partial\theta}\right|_{\underline{\Omega}}
\end{aligned} \tag{2.56}
$$

Then

$$
\boldsymbol{q} = -k(\theta_{\underline{\Omega}}, {}^{q}\underset{\sim}{\underline{I}}|_{\underline{\Omega}})\boldsymbol{g} - k_1(\theta_{\underline{\Omega}}, {}^{q}\underset{\sim}{\underline{I}}|_{\underline{\Omega}})(\boldsymbol{g}\cdot\boldsymbol{g})\boldsymbol{g} - k_2(\theta_{\underline{\Omega}}, {}^{q}\underset{\sim}{\underline{I}}|_{\underline{\Omega}})(\theta - \theta_{\underline{\Omega}})\boldsymbol{g} \tag{2.57}
$$

This is the simplest possible constitutive theory based on theory of generators and invariants using (2.50). The only assumption in this theory beyond (2.50) is the truncation of the Taylor series in (2.52) beyond linear terms in ${}^q\underset{\sim}{I}$ and θ.

2.4.2.2 Approach II

In this case, we consider

$$\boldsymbol{q} = \boldsymbol{q}(\boldsymbol{\varepsilon}_{[0]}, \theta, \boldsymbol{g}) \tag{2.58}$$

As shown in (2.11), this is a more general case due to dependence of $\boldsymbol{q}$ on $\boldsymbol{g}$, θ, as well as $\boldsymbol{\varepsilon}_{[0]}$. $\boldsymbol{q}$ is a tensor of rank one, whereas $\boldsymbol{\varepsilon}_{[0]}$, $\boldsymbol{g}$, and θ are symmetric tensor of rank two, tensor of rank one, and tensor of rank zero respectively. Justification for retaining $\boldsymbol{\varepsilon}_{[0]}$ as an argument tensor of $\boldsymbol{q}$ (over and beyond the principle of equipresence) will be discussed after we present the details of the constitutive theory for $\boldsymbol{q}$ based on (2.58) by using the theory of generators and invariants. The combined generators of rank one of the argument tensors $\boldsymbol{\varepsilon}_{[0]}$, $\boldsymbol{g}$, and θ are

$$\{{}^q\underset{\sim}{G}^1\} = \boldsymbol{g};\ \{{}^q\underset{\sim}{G}^2\} = \boldsymbol{\varepsilon}_{[0]}\boldsymbol{g};\ \{{}^q\underset{\sim}{G}^3\} = (\boldsymbol{\varepsilon}_{[0]})^2\boldsymbol{g} \tag{2.59}$$

The combined invariants of the argument tensors $\boldsymbol{\varepsilon}_{[0]}$, $\boldsymbol{g}$, and θ are

$$\begin{aligned} &{}^q\underset{\sim}{I}^1 = \operatorname{tr}\boldsymbol{\varepsilon}_{[0]}; \quad {}^q\underset{\sim}{I}^2 = \operatorname{tr}\left((\boldsymbol{\varepsilon}_{[0]})^2\right); \quad {}^q\underset{\sim}{I}^3 = \operatorname{tr}\left((\boldsymbol{\varepsilon}_{[0]})^3\right); \\ &{}^q\underset{\sim}{I}^4 = \boldsymbol{g}\cdot\boldsymbol{g}; \quad {}^q\underset{\sim}{I}^5 = \boldsymbol{g}\cdot\boldsymbol{\varepsilon}_{[0]}\boldsymbol{g}; \quad {}^q\underset{\sim}{I}^6 = \boldsymbol{g}\cdot(\boldsymbol{\varepsilon}_{[0]})^2\boldsymbol{g} \end{aligned} \tag{2.60}$$

We note that for ${}^q\underset{\sim}{I}^j;\ j = 1,2,3$, we could have also used $I_{\boldsymbol{\varepsilon}_{[0]}}$, $II_{\boldsymbol{\varepsilon}_{[0]}}$, and $III_{\boldsymbol{\varepsilon}_{[0]}}$. As the two sets of invariants are related, the resulting constitutive theory remains unaffected. Using the generators in (2.59), we can write

$$\boldsymbol{q} = -\sum_{i=1}^{3} {}^q\underset{\sim}{\alpha}^i\{{}^q\underset{\sim}{G}^i\} \tag{2.61}$$

The coefficients ${}^q\underset{\sim}{\alpha}^i$ in the linear combination are functions of the invariants ${}^q\underset{\sim}{I}^j;\ j =$

$1, 2, \ldots, 6$ and θ in the current configuration. To determine the material coefficients from ${}^q\widetilde{\alpha}^i;\ i = 1,2,3$ in (2.61), we consider Taylor series expansion of ${}^q\widetilde{\alpha}^i;\ i = 1,2,3$ in ${}^q\underset{\sim}{I}^j;\ j = 1, 2, \ldots, 6$ and θ about a known configuration $\underline{\Omega}$ and retain only up to linear terms in the invariants and θ.

$$
{}^q\widetilde{\alpha}^i = {}^q\widetilde{\alpha}^i|_{\underline{\Omega}} + \sum_{j=1}^{6} \left. \frac{\partial\, {}^q\widetilde{\alpha}^i}{\partial\, {}^q\underset{\sim}{I}^j} \right|_{\underline{\Omega}} \left({}^q\underset{\sim}{I}^j - ({}^q\underset{\sim}{I}^j)_{\underline{\Omega}} \right) + \left. \frac{\partial\, {}^q\widetilde{\alpha}^i}{\partial \theta} \right|_{\underline{\Omega}} (\theta - \theta_{\underline{\Omega}});\ i = 1,2,3 \tag{2.62}
$$

${}^q\widetilde{\alpha}^i|_{\underline{\Omega}}$, $\left.\frac{\partial\, {}^q\widetilde{\alpha}^i}{\partial\, {}^q\underset{\sim}{I}^j}\right|_{\underline{\Omega}}$; $j = 1, 2, \ldots, 6$ and $\left.\frac{\partial\, {}^q\widetilde{\alpha}^i}{\partial \theta}\right|_{\underline{\Omega}}$; $i = 1,2,3$ are functions of $\theta|_{\underline{\Omega}}$, ${}^q\underset{\sim}{I}^j|_{\underline{\Omega}}$; $j = 1, 2, \ldots, 6$ whereas ${}^q\widetilde{\alpha}^i = {}^q\widetilde{\alpha}^i(\theta|_{\underline{\Omega}}, {}^q\underset{\sim}{I}^j|_{\underline{\Omega}}, \theta, {}^q\underset{\sim}{I}^j);\ j = 1, 2, \ldots, 6$. We substitute from (2.62) into (2.61)

$$
\boldsymbol{q} = -\sum_{i=1}^{3} \left({}^q\widetilde{\alpha}^i|_{\underline{\Omega}} + \sum_{j=1}^{6} \left. \frac{\partial\, {}^q\widetilde{\alpha}^i}{\partial\, {}^q\underset{\sim}{I}^j} \right|_{\underline{\Omega}} \left({}^q\underset{\sim}{I}^j - ({}^q\underset{\sim}{I}^j)_{\underline{\Omega}} \right) + \left. \frac{\partial\, {}^q\widetilde{\alpha}^i}{\partial \theta} \right|_{\underline{\Omega}} (\theta - \theta_{\underline{\Omega}}) \right) \{{}^q\underset{\sim}{G}^i\} \tag{2.63}
$$

Using (2.63), we collect the coefficients (those defined in the configuration $\underline{\Omega}$) of $\{{}^q\underset{\sim}{G}^i\}$, ${}^q\widetilde{I}^j\{{}^q\underset{\sim}{G}^i\};\ i = 1,2,3;\ j = 1, 2, \ldots, 6$ and define

$$
\begin{aligned}
{}^q\widetilde{a}_i &= {}^q\widetilde{\alpha}^i|_{\underline{\Omega}} + \sum_{j=1}^{6} \left. \frac{\partial\, {}^q\widetilde{\alpha}^i}{\partial\, {}^q\underset{\sim}{I}^j} \right|_{\underline{\Omega}} \left({}^q\underset{\sim}{I}^j - ({}^q\underset{\sim}{I}^j)_{\underline{\Omega}} \right);\ i = 1,2,3 \\
{}^q\widetilde{b}_{ij} &= \left. \frac{\partial\, {}^q\widetilde{\alpha}^i}{\partial\, {}^q\underset{\sim}{I}^j} \right|_{\underline{\Omega}};\ i = 1,2,3;\ j = 1, 2, \ldots, 6 \\
{}^q\widetilde{c}_i &= \left. \frac{\partial\, {}^q\widetilde{\alpha}^i}{\partial \theta} \right|_{\underline{\Omega}};\ i = 1,2,3
\end{aligned} \tag{2.64}
$$

${}^q\widetilde{a}_i$, ${}^q\widetilde{b}_{ij}$, and ${}^q\widetilde{c}_i$ are the material coefficients defined in the known configuration $\underline{\Omega}$. This constitutive theory for $\boldsymbol{q}$ uses the full set of argument tensors and integrity and hence is complete. Unfortunately, it requires too many material coefficients (twenty-four).

Remarks. (1) With some assumptions, this constitutive theory for heat vector can be simplified to yield approximate constitutive theories in which the material coefficients may not be as many as in (2.64). This will undoubtedly limit the physics. If we limit the constitutive theory such that we neglect generator $\{{}^q\underset{\sim}{G}^3\} = (\boldsymbol{\varepsilon}_{[0]})^2\boldsymbol{g}$ and the invariants ${}^q\underset{\sim}{I}^2$, ${}^q\underset{\sim}{I}^3$, and ${}^q\underset{\sim}{I}^6$, the constitutive theory in (2.64)reduces to

$$
\begin{aligned}
\boldsymbol{q} = -{}^q\widetilde{a}_1\boldsymbol{g} - {}^q\widetilde{a}_2\boldsymbol{\varepsilon}_{[0]}\boldsymbol{g} - {}^q\widetilde{b}_{11}\big(\operatorname{tr}\boldsymbol{\varepsilon}_{[0]}\big)\boldsymbol{g}& \\
- {}^q\widetilde{b}_{14}(\boldsymbol{g}\cdot\boldsymbol{g})\boldsymbol{g} - {}^q\widetilde{b}_{15}(\boldsymbol{g}\cdot\boldsymbol{\varepsilon}_{[0]}\boldsymbol{g})\boldsymbol{g}& \\
- {}^q\widetilde{b}_{21}\big(\operatorname{tr}\boldsymbol{\varepsilon}_{[0]}\big)\big(\boldsymbol{\varepsilon}_{[0]}\boldsymbol{g}\big) - {}^q\widetilde{b}_{24}(\boldsymbol{g}\cdot\boldsymbol{g})\big(\boldsymbol{\varepsilon}_{[0]}\boldsymbol{g}\big)& \\
- {}^q\widetilde{b}_{25}(\boldsymbol{g}\cdot\boldsymbol{\varepsilon}_{[0]}\boldsymbol{g})\big(\boldsymbol{\varepsilon}_{[0]}\boldsymbol{g}\big) - {}^q\widetilde{c}_1(\theta-\theta_{\underline{\Omega}})\boldsymbol{g}& \\
- {}^q\widetilde{c}_2(\theta-\theta_{\underline{\Omega}})\boldsymbol{\varepsilon}_{[0]}\boldsymbol{g}& \quad (2.65)
\end{aligned}
$$

This constitutive theory still requires ten material coefficients. If we further assume that the constitutive theory is linear in the components of $\boldsymbol{\varepsilon}_{[0]}$, then the terms containing material coefficients ${}^q\widetilde{b}_{21}$ and ${}^q\widetilde{b}_{25}$ can be removed from (2.65).

$$
\begin{aligned}
\boldsymbol{q} = -{}^q\widetilde{a}_1\boldsymbol{g} - {}^q\widetilde{a}_2\boldsymbol{\varepsilon}_{[0]}\boldsymbol{g} - {}^q\widetilde{b}_{11}\big(\operatorname{tr}\boldsymbol{\varepsilon}_{[0]}\big)\boldsymbol{g}& \\
- {}^q\widetilde{b}_{14}(\boldsymbol{g}\cdot\boldsymbol{g})\boldsymbol{g} - {}^q\widetilde{b}_{15}(\boldsymbol{g}\cdot\boldsymbol{\varepsilon}_{[0]}\boldsymbol{g})\boldsymbol{g}& \\
- {}^q\widetilde{b}_{24}(\boldsymbol{g}\cdot\boldsymbol{g})\big(\boldsymbol{\varepsilon}_{[0]}\boldsymbol{g}\big) - {}^q\widetilde{c}_1(\theta-\theta_{\underline{\Omega}})\boldsymbol{g}& \\
- {}^q\widetilde{c}_2(\theta-\theta_{\underline{\Omega}})\boldsymbol{\varepsilon}_{[0]}\boldsymbol{g}& \quad (2.66)
\end{aligned}
$$

This constitutive theory requires eight material coefficients.

(2) If we remove the dependence of $\boldsymbol{q}$ on $\boldsymbol{\varepsilon}_{[0]}$ in (2.66), then

$$
\boldsymbol{q} = -{}^q\widetilde{a}_1\boldsymbol{g} - {}^q\widetilde{b}_{14}(\boldsymbol{g}\cdot\boldsymbol{g})\boldsymbol{g} - {}^q\widetilde{c}_1(\theta-\theta_{\underline{\Omega}})\boldsymbol{g} \quad (2.67)
$$

This constitutive theory for $\boldsymbol{q}$ is the same as derived earlier (equation (2.57)). The coefficients in (2.67) are functions of $\theta|_{\underline{\Omega}}$ and $(\boldsymbol{g}\cdot\boldsymbol{g})_{\underline{\Omega}}$.

(3) To demonstrate the influence of strain field on heat conduction, we reduce the consti-

tutive theory (2.66) to a most simplified theory by considering $\boldsymbol{q}$ to be a linear function of the components of $\boldsymbol{g}$ as well as $\boldsymbol{\varepsilon}_{[0]}$.

$$\begin{gathered}\boldsymbol{q} = -\, {}^q\widetilde{a}_1\boldsymbol{g} - {}^q\widetilde{a}_2\boldsymbol{\varepsilon}_{[0]}\boldsymbol{g} - {}^q\widetilde{b}_{11}\big(\operatorname{tr}\boldsymbol{\varepsilon}_{[0]}\big)\boldsymbol{g} - \\ {}^q\widetilde{c}_1(\theta-\theta_{\underline{\Omega}})\boldsymbol{g} - {}^q\widetilde{c}_2(\theta-\theta_{\underline{\Omega}})\boldsymbol{\varepsilon}_{[0]}\boldsymbol{g}\end{gathered} \tag{2.68}$$

If we neglect $(\theta-\theta_{\underline{\Omega}})$ terms in (2.68), then we obtain

$$\boldsymbol{q} = -{}^q\widetilde{a}_1\boldsymbol{g} - {}^q\widetilde{a}_2\boldsymbol{\varepsilon}_{[0]}\boldsymbol{g} - {}^q\widetilde{b}_{11}\big(\operatorname{tr}\boldsymbol{\varepsilon}_{[0]}\big)\boldsymbol{g} \tag{2.69}$$

The material coefficients in (2.69) are functions of $\theta_{\underline{\Omega}}$, $(\boldsymbol{g}\cdot\boldsymbol{g})_{\underline{\Omega}}$, $(\operatorname{tr}\boldsymbol{\varepsilon}_{[0]})_{\underline{\Omega}}$, and $(\boldsymbol{g}\cdot\boldsymbol{\varepsilon}_{[0]}\boldsymbol{g})_{\underline{\Omega}}$. This constitutive theory now only requires three material coefficients. We can also write (2.69) as

$$\boldsymbol{q} = -{}^q\widetilde{a}_1\boldsymbol{g} - \big({}^q\widetilde{a}_2\boldsymbol{\varepsilon}_{[0]} + {}^q\widetilde{b}_{11}(\operatorname{tr}\boldsymbol{\varepsilon}_{[0]})\boldsymbol{I}\big)\boldsymbol{g} \tag{2.70}$$

If we let ${}^q\widetilde{a}_1 = k$, ${}^q\widetilde{a}_2 = k_1$, and ${}^q\widetilde{b}_{11} = k_2$, where k, k_1, and k_2 are positive material coefficients, then (2.70) can be written as

$$\boldsymbol{q} = -\big(k\boldsymbol{I} - k_1\boldsymbol{\varepsilon}_{[0]} - k_2(\operatorname{tr}\boldsymbol{\varepsilon}_{[0]})\boldsymbol{I}\big)\boldsymbol{g} = -\underset{\sim}{\boldsymbol{k}}\boldsymbol{g} \tag{2.71}$$

in which $\underset{\sim}{\boldsymbol{k}}$ is the effective conductivity matrix in the presence of strain field. The coefficient of $\boldsymbol{g}$ in the second term on the right side of (2.70) is the influence of the strain field on the heat conduction (in the most simplified form of the constitutive theory for $\boldsymbol{q}$).

(4) From (2.71) for the 1-D case (in $\mathbb{R}^1$), we can write

$$q_{x_1} = -\big(k - (k_1+k_2)(\varepsilon)_{x_1x_1}\big)\frac{\partial\theta}{\partial x_1} = -\underset{\sim}{k}\frac{\partial\theta}{\partial x_1} \tag{2.72}$$

From (2.72) we clearly see that compression (negative $(\varepsilon)_{x_1x_1}$) enhances heat conduction due to increased $\underset{\sim}{k}$. This of course is due to faster vibrational energy transfer at lower scale (mode of heat transfer) due to reduced mean free path between the molecules due to compression. On the other hand, tension (positive $(\varepsilon)_{x_1x_1}$) increases mean free path between the molecules, hence the vibrational energy transfer between the molecules is reduced compared to the unstressed state. In tension, effective $\underset{\sim}{k}$ is obviously reduced. We remark that the influence of strain field on heat transfer is most significant under high compression or tension as it influences the mean free path significantly. All matter in reality is compressible; however, the degree of compressibility may vary depending upon the matter and the application.

Chapter 3

Ordered rate constitutive theories in Lagrangian description for thermoviscoelastic solids without memory

3.1 Introduction

In the work presented here, we consider the derivation of ordered rate constitutive theories for homogeneous, isotropic thermoviscoelastic solids without memory experienced from the deformation. To ensure thermodynamic equilibrium during the evolution, the rate constitutive theories are derived using the second law of thermodynamics. If the entropy inequality is expressed in terms of the Helmholtz free energy density Φ, then Φ, entropy density η, heat vector $\boldsymbol{q}$, and second Piola-Kirchhoff stress tensor $\boldsymbol{\sigma}^{[0]}$ are established as possible choices of dependent variables in the constitutive theory. At the onset of the development, we choose $\boldsymbol{\sigma}^{[0]}$ and Green's strain tensor $\boldsymbol{\varepsilon}$ (or $\boldsymbol{\varepsilon}_{[0]}$) as a conjugate pair.

The arguments of the dependent variables are established beginning with $\boldsymbol{J}$, the fundamental measure of deformation, and also including $\boldsymbol{J}_{[i]};\ i = 1, 2, \ldots, n$ in addition to $\boldsymbol{J}$ as arguments of the dependent variables. At the very least, $\boldsymbol{J}_{[1]}$ is necessitated as an argument to introduce a dissipation mechanism in the constitutive theories. The introduction of $\boldsymbol{J}_{[i]};\ i = 1, 2, \ldots, n$, compared to only $\boldsymbol{J}_{[1]}$, is a generalization. By following the axiom of frame invariance, a polar decomposition and dependence of $\boldsymbol{S}_r$ on $\boldsymbol{\varepsilon}_{[0]}$ and its generalization eventually leads to $\boldsymbol{\varepsilon}_{[0]}$ and $\boldsymbol{\varepsilon}_{[i]};\ i = 1, 2, \ldots, n$ as argument tensors instead of $\boldsymbol{J}$ and $\boldsymbol{J}_{[i]}$; $i = 1, 2, \ldots, n$ for the dependent variables Φ, η, $\boldsymbol{\sigma}^{[0]}$, and $\boldsymbol{q}$. In addition, we also consider $\boldsymbol{g}$ and θ as argument tensors of all dependent variables. Using $\Phi = \Phi(\boldsymbol{\varepsilon}_{[i]};\ i = 0, 1, \ldots, n, \boldsymbol{g}, \theta)$, when $\dot{\Phi}$ (obtained using the chain rule of differentiation) is substituted in the entropy inequality, we can conclude Φ, $\boldsymbol{\sigma}^{[0]}$, and $\boldsymbol{q}$ are the dependent variables in the constitutive theories.

From the conditions resulting from the entropy inequality, there is no mechanism to derive the constitutive theories for $\boldsymbol{\sigma}^{[0]}$ and $\boldsymbol{q}$. Upon decomposition of stress $\boldsymbol{\sigma}^{[0]}$ into equilibrium stress ${}_{e}\boldsymbol{\sigma}^{[0]}$ and deviatoric stress ${}_{d}\boldsymbol{\sigma}^{[0]}$, the conditions resulting from the entropy inequality are used to establish the constitutive theory for ${}_{e}\boldsymbol{\sigma}^{[0]}$ as thermodynamic pressure as a function of density and temperature for compressible matter and mechanical pressure as a function of temperature (after introducing the incompressibility constraint in the entropy inequality). After substituting the stress decomposition into the entropy inequality, the resulting conditions also require that the work expended due to ${}_{d}\boldsymbol{\sigma}^{[0]}$ be positive, but they provide no mechanism for deriving constitutive theories for ${}_{d}\boldsymbol{\sigma}^{[0]}$. We also have $q_i g_i \leq 0$ from the entropy inequality, which can be used to derive the constitutive theory for $\boldsymbol{q}$.

In this work, we derive the constitutive theory for ${}_{d}\boldsymbol{\sigma}^{[0]}$ using the theory of generators and invariants [31]. The constitutive theories for $\boldsymbol{q}$ are also derived using the theory of generators and invariants. These theories are of order n, since these use stress rates of up to order n, hence the name, ordered rate constitutive theories.

The constitutive theories for $\boldsymbol{q}$ are also derived using the conditions resulting from the entropy inequality and using the theory of generators and invariants with reduced argument tensors. Many simplified forms of the rate theories of order n are considered and compared with currently used theories such as the Kelvin-Voigt model, models based on velocity dependent dissipation, etc. Numerical studies are presented using space-time *hpk* finite element processes [52–57] for 1-d wave propagation using (i) the constitutive theories developed here and (ii) velocity-dependent dissipation, and the resulting evolutions are compared and discussed. In this work, we show that the mechanism of dissipation is due to each strain rate $\boldsymbol{\varepsilon}_{[i]}$; $i = 1, 2, \ldots, n$. Thus, at the very least, we must include $\boldsymbol{\varepsilon}_{[1]}$ as an argument tensor of $\boldsymbol{\sigma}^{[0]}$ and $\boldsymbol{q}$ in addition to $\boldsymbol{\varepsilon}_{[0]}$, $\boldsymbol{g}$, and θ to have a dissipative mechanism in the resulting constitutive theories.

3.2 Second law of thermodynamics using Φ and conjugate pair (σ^*, $\dot{\boldsymbol{J}}$), dependent variables in the rate constitutive theories, and their arguments

The fundamental principles of continuum mechanics: conservation of mass, balance of momenta, and the first and second laws of thermodynamics must be satisfied by all deforming matter during evolution to ensure thermodynamic equilibrium. Since the first three principles are independent of the constitution of the matter, the second law of thermodynamics, *i.e.*, entropy or Clausius-Duhem inequality must provide a basis or mechanism for describing the constitution of the matter. The entropy inequality in Lagrangian description can be derived using Helmholtz free energy density Φ and conjugate pairs $\boldsymbol{\sigma}^*$, $\dot{\boldsymbol{J}}$, or $\boldsymbol{\sigma}^{[0]}$, $\dot{\boldsymbol{\varepsilon}}$ [1–3]. $\boldsymbol{\sigma}^*$ is the first Piola- Kirchhoff stress tensor, $\dot{\boldsymbol{J}}$ is the material derivative of the Jacobian of deformation $J_{ij} = \frac{\partial \bar{x}_i}{\partial x_j}$ where x_k and $\bar{x}_k$ are undeformed and deformed coordinates, $\boldsymbol{\sigma}^{[0]}$ is the second Piola-Kirchhoff stress tensor derived using contravariant Cauchy stress tensor, and $\dot{\boldsymbol{\varepsilon}}$ is the material derivative of the Green's strain tensor. In this paper, we utilize both forms of the entropy inequality. Both forms of the entropy inequality are exactly equivalent as the conjugate pairs $\boldsymbol{\sigma}^*, \dot{\boldsymbol{J}}$ and $\boldsymbol{\sigma}^{[0]}, \dot{\boldsymbol{\varepsilon}}$ are transformable from each other. Thus, the choice of one or the other does not matter. At this point we consider entropy inequality in Φ using conjugate pair $\boldsymbol{\sigma}^*$, $\dot{\boldsymbol{J}}$

$$\rho_0(\dot{\Phi} + \eta\dot{\theta}) + \frac{|J| q_i g_i}{\theta} - \sigma^*_{ik}\dot{J}_{ik} \leq 0 \tag{3.1}$$

in which ρ_0 is the density in the undeformed configuration (also used as reference configuration). η is the entropy density, θ is the absolute temperature, $|J|$ is the determinant of the Jacobian of deformation, $\boldsymbol{q}$ is the heat vector, and $\boldsymbol{g}$ is the temperature gradient.

From the balance of momenta and the first law of thermodynamics, in which we assume the existence of a stress field and heat vector, we conclude that the stress tensor and heat vector must be dependent variables in the constitutive theories. This is also supported by the entropy inequality (3.1). In addition, from (3.1), we note that Φ and η must also be considered as dependent variables in the constitutive theories. Thus, at this stage, we have $\boldsymbol{\sigma}^*$, $\boldsymbol{q}$, Φ, and η as possible dependent variables in the constitutive theories, keeping in mind that some of these may be eliminated at a later stage if so warranted due to some other considerations. θ, $\boldsymbol{J}$, $\dot{\boldsymbol{J}}$, and $\boldsymbol{g}$ cannot be dependent variables in the constitutive theories as either they are self observable or can be defined using self observable quantities such as temperature and material point displacements.

The Jacobian of deformation $\boldsymbol{J}$ is the fundamental measure of deformation, hence must be considered as an argument of all dependent variables in the constitutive theories. Since we are considering rate theories, all dependent variables in the constitutive theories must also exhibit dependence on the material derivative of $\boldsymbol{J}$. We note that $\boldsymbol{J}_{[i]};\ i = 1, 2, \dots, n$, the material derivatives of $\boldsymbol{J}$ up to order n are fundamental kinematic measures of rates of $\boldsymbol{J}$ of various orders up to n, *i.e.*, these are measures of rates of $\boldsymbol{J}$ up to order n and are linearly independent. Hence, at the onset of development of rate theories, we also consider $\boldsymbol{J}_{[i]};\ i = 1, 2, \dots, n$ as arguments of all dependent variables in the constitutive theories. Additionally, we also consider θ and $\boldsymbol{g}$ as arguments of $\boldsymbol{\sigma}^*$, $\boldsymbol{q}$, Φ, and η. Thus, now we have identified the dependent variables and their arguments in the development of the rate constitutive theories, and we can write:

$$
\begin{aligned}
\Phi &= \Phi\big(\boldsymbol{J}, \boldsymbol{J}_{[i]};\ i = 1, 2, \ldots, n, \boldsymbol{g}, \theta\big) \\
\boldsymbol{\sigma}^* &= \boldsymbol{\sigma}^*\big(\boldsymbol{J}, \boldsymbol{J}_{[i]};\ i = 1, 2, \ldots, n, \boldsymbol{g}, \theta\big) \\
\boldsymbol{q} &= \boldsymbol{q}\big(\boldsymbol{J}, \boldsymbol{J}_{[i]};\ i = 1, 2, \ldots, n, \boldsymbol{g}, \theta\big) \\
\eta &= \eta\big(\boldsymbol{J}, \boldsymbol{J}_{[i]};\ i = 1, 2, \ldots, n, \boldsymbol{g}, \theta\big)
\end{aligned}
\tag{3.2}
$$

We note $\boldsymbol{J}_{[1]}$, the first material derivative of $\boldsymbol{J}$, is sometimes also denoted as $\dot{\boldsymbol{J}}$, as in (3.1) (to conform to commonly used notation), but is used as $\boldsymbol{J}_{[1]}$ in (3.2) for compact and consistent presentation. Now, since the arguments of Φ are defined in (3.2), we can obtain a more explicit form of $\dot{\Phi}$ using Φ in (3.2)

$$
\dot{\Phi} = \frac{\partial \Phi}{J_{kl}} \dot{J}_{kl} + \sum_{i=1}^{n} \frac{\partial \Phi}{\partial (J_{[i]})_{kl}} (\dot{J}_{[i]})_{kl} + \frac{\partial \Phi}{\partial \theta} \dot{\theta} + \frac{\partial \Phi}{\partial g_i} \dot{g}_i \tag{3.3}
$$

Substituting from (3.3) into (3.1) and collecting coefficients

$$
\left(\rho_0 \frac{\partial \Phi}{\partial J_{kl}} - \sigma^*_{lk} \right) \dot{J}_{kl} + \rho_0 \left(\frac{\partial \Phi}{\partial \theta} + \eta \right) \dot{\theta} + \sum_{i=1}^{n} \rho_0 \frac{\partial \Phi}{\partial (J_{[i]})_{kl}} (\dot{J}_{[i]})_{kl} + \rho_0 \frac{\partial \Phi}{\partial g_i} \dot{g}_i + \frac{|J| q_i g_i}{\theta} \leq 0 \tag{3.4}
$$

In order for (3.4) to hold for arbitrary but admissible $\dot{\boldsymbol{J}}_{[i]};\ i = 1, 2, \ldots, n$, $\dot{\boldsymbol{g}}$, and $\dot{\theta}$, the following must hold:

$$
\begin{aligned}
\left(\rho_0 \frac{\partial \Phi}{\partial J_{kl}} - \sigma^*_{lk} \right) \dot{J}_{kl} + \frac{|J| q_i g_i}{\theta} &\leq 0 \\
\rho_0 \left(\frac{\partial \Phi}{\partial \theta} + \eta \right) &= 0 \\
\rho_0 \frac{\partial \Phi}{\partial (J_{[i]})_{kl}} &= 0;\ i = 1, 2, \ldots, n \\
\rho_0 \frac{\partial \Phi}{\partial g_i} &= 0
\end{aligned}
\tag{3.5}
$$

Since ρ_0 is constant, ρ_0 in the last three equations in (3.5) can be dropped, and we have:

$$\left(\rho_0 \frac{\partial \Phi}{\partial J_{kl}} - \sigma^*_{lk}\right) \dot{J}_{kl} + \frac{|J| q_i g_i}{\theta} \leq 0 \tag{3.6}$$

$$\left(\frac{\partial \Phi}{\partial \theta} + \eta\right) = 0 \tag{3.7}$$

$$\frac{\partial \Phi}{\partial (J_{[i]})_{kl}} = 0;\ i = 1, 2, \ldots, n \tag{3.8}$$

$$\frac{\partial \Phi}{\partial g_i} = 0 \tag{3.9}$$

Remarks. (1) Equation (3.8) implies that Φ is not a function of $\boldsymbol{J}_{[i]};\ i = 1, 2, \ldots, n$.

(2) Equation (3.9) implies that Φ is not a function of $\boldsymbol{g}$ either.

(3) From equation (3.7), we have $\eta = -\frac{\partial \Phi}{\partial \theta}$, hence η is deterministic from Φ, therefore η is not a dependent variable in the constitutive theories.

(4) The inequality in (3.6) is essential in the form it is stated. For example

$$\rho_0 \frac{\partial \Phi}{\partial J_{kl}} - \sigma^*_{lk} = 0 \text{ and } \frac{|J| q_i g_i}{\theta} \leq 0 \tag{3.10}$$

is inappropriate due to the fact that these imply that $\boldsymbol{\sigma}^*$ is not a function of $\boldsymbol{J}_{[i]};\ i = 1, 2, \ldots, n$, as Φ is not a function of $\boldsymbol{J}_{[i]};\ i = 1, 2, \ldots, n$, which is contrary to (3.2). We note that (3.6) in the stated form is unable to provide further details or mechanism for deriving constitutive theories for $\boldsymbol{\sigma}^*$ and $\boldsymbol{q}$.

(5) Based on remarks 1-4, (3.2) reduces to

$$\begin{aligned} \Phi &= \Phi(\boldsymbol{J}, \theta) \\ \boldsymbol{\sigma}^* &= \boldsymbol{\sigma}^*(\boldsymbol{J}, \boldsymbol{J}_{[i]};\ i = 1, 2, \ldots, n, \boldsymbol{g}, \theta) \\ \boldsymbol{q} &= \boldsymbol{q}(\boldsymbol{J}, \boldsymbol{J}_{[i]};\ i = 1, 2, \ldots, n, \boldsymbol{g}, \theta) \end{aligned} \tag{3.11}$$

Thus the constitutive theories for these solids reduce to determination of dependence of $\boldsymbol{\sigma}^*$ and $\boldsymbol{q}$ on the deformation using (3.11).

3.2.1 Stress decomposition

To proceed further using (3.6) resulting from the entropy inequality, we perform decomposition of $\boldsymbol{\sigma}^*$ into equilibrium stress (or mean normal stress in case of incompressible matter) ${}_e\boldsymbol{\sigma}^*$ and deviatoric stress ${}_d\boldsymbol{\sigma}^*$

$$\boldsymbol{\sigma}^* = {}_e\boldsymbol{\sigma}^* + {}_d\boldsymbol{\sigma}^* \tag{3.12}$$

in which we have the following

$${}_e\boldsymbol{\sigma}^* = {}_e\boldsymbol{\sigma}^*\big(\boldsymbol{J},\theta\big) \tag{3.13}$$

$${}_d\boldsymbol{\sigma}^* = {}_d\boldsymbol{\sigma}^*\big(\boldsymbol{J},\boldsymbol{J}_{[i]};\ i=1,2,\ldots,n,\boldsymbol{g},\theta\big) \tag{3.14}$$

That is, ${}_e\boldsymbol{\sigma}^*$ is not a function of $\boldsymbol{J}_{[i]};\ i=1,2,\ldots,n$ and $\boldsymbol{g}$.

Substituting (3.12) in (3.6), we obtain

$$\left(\rho_0\frac{\partial\Phi}{\partial J_{ik}} - {}_e\sigma^*_{ki} - {}_d\sigma^*_{ki}\right)\dot{J}_{ik} + \frac{|J|q_i g_i}{\theta} \leq 0 \tag{3.15}$$

or

$$\left(\rho_0\frac{\partial\Phi}{\partial J_{ik}} - {}_e\sigma^*_{ki}\right)\dot{J}_{ik} - {}_d\sigma^*_{ki}\dot{J}_{ik} + \frac{|J|q_i g_i}{\theta} \leq 0 \tag{3.16}$$

Since Φ is not a function of $\boldsymbol{J}_{[i]};\ i=1,2,\ldots,n$ (remark 1) and neither is ${}_e\boldsymbol{\sigma}^*$ (in (3.13)), ${}_e\boldsymbol{\sigma}^*$ must be derivable from

$${}_e\sigma^*_{ki} = \rho_0\frac{\partial\Phi}{\partial J_{ik}} \text{ or } \left({}_e\boldsymbol{\sigma}^*\right)^T = \rho_0\frac{\partial\Phi}{\partial\boldsymbol{J}} \tag{3.17}$$

Using (3.17), the inequality (3.16) reduces to

$$-\,{}_{d}\sigma^{*}_{ki}\dot{J}_{ik} + \frac{|J|q_i g_i}{\theta} \le 0 \tag{3.18}$$

If we assume

$$\frac{|J|q_i g_i}{\theta} \le 0, \tag{3.19}$$

then (3.18) is satisfied if

$${}_{d}\sigma^{*}_{ki}\dot{J}_{ik} > 0 \tag{3.20}$$

Equation (3.20) requires that work expended due to deviatoric stress must be positive. In view of (3.17), the stress decomposition (3.12) can be written as

$$\sigma^{*}_{ij} = \rho_0 \frac{\partial \Phi}{\partial J_{ji}} + {}_{d}\sigma^{*}_{ij}\big(\boldsymbol{J}, \boldsymbol{J}_{[i]};\ i = 1, 2, \ldots, n, \boldsymbol{g}, \theta\big) \tag{3.21}$$

Additionally, we have

$$\Phi = \Phi\big(\boldsymbol{J}, \theta\big) \tag{3.22}$$

$$\boldsymbol{q} = \boldsymbol{q}\big(\boldsymbol{J}, \boldsymbol{J}_{[i]};\ i = 1, 2, \ldots, n, \boldsymbol{g}, \theta\big) \tag{3.23}$$

Remarks. When $\boldsymbol{J}_{[i]};\ i = 1, 2, \ldots, n$ (all or any) are arguments of the dependent variables in the Constitutive theories, we note that:

(1) The entropy inequality requires decomposition of the stress tensor into equilibrium and deviatoric stress tensors.

(2) Based on the conditions (3.17) resulting from the entropy inequality (after stress decomposition), the equilibrium stress is deterministic from the Helmholtz free energy density, but the constitutive theory for deviatoric stress tensor is not. Thus in the case of rate constitutive theories, the entropy inequality (with stress decomposition) can only take us as far as (3.19) – (3.23). The constitutive theories for ${}_{d}\boldsymbol{\sigma}^{*}$ and $\boldsymbol{q}$ must

satisfy (3.19) and (3.20) while ${}_e\boldsymbol{\sigma}^*$ must be derived using (3.17).

(3) The arguments of the dependent variables need further considerations: (i) first, due to frame invariance requirement, as $\boldsymbol{J}$ and $\boldsymbol{J}_{[i]};\ i = 1,2,\dots,n$ are not frame invariant. (ii) Since the conditions resulting from the entropy inequality do not provide a mechanism for deriving constitutive theory for ${}_d\boldsymbol{\sigma}^*$, we shall consider theory of generators and invariants [1–3] for deriving constitutive theory for ${}_d\boldsymbol{\sigma}^*$, which requires arguments of ${}_d\boldsymbol{\sigma}^*$ to be tensors. $\boldsymbol{g}$ and θ are tensors of rank one and zero, however $\boldsymbol{J}$ and $\boldsymbol{J}_{[i]};\ i = 1,2,\dots,n$ are not tensors. Hence $\boldsymbol{J}$ and $\boldsymbol{J}_{[i]};\ i = 1,2,\dots,n$ must be replaced by equivalent measures that are tensors.

3.3 Further considerations on dependent variables and their arguments

Due to the principle of frame invariance, the rotation part of $\boldsymbol{J}$ and $\boldsymbol{J}_{[i]};\ i = 1,2,\dots,n$ cannot be part of the constitutive theories. We consider polar decomposition of $\boldsymbol{J}$:

$$\boldsymbol{J} = \boldsymbol{R}\boldsymbol{S}_r = \boldsymbol{S}_l\boldsymbol{R} \tag{3.24}$$

In (3.24), $\boldsymbol{S}_r$ and $\boldsymbol{S}_l$ are right and left stretch tensors that are symmetric and positive definite, and $\boldsymbol{R}$ is the rotation matrix, hence defines rotation and therefore cannot be part of the constitutive theory. Thus, in (3.24), we must replace $\boldsymbol{J}$ by $\boldsymbol{S}_r$ (or $\boldsymbol{S}_l$ if so desired). However, $\boldsymbol{S}_r$ can be expressed in terms of Green's strain tensor $\boldsymbol{\varepsilon}$ or $\boldsymbol{\varepsilon}_{[0]}$ (material derivative of order zero)

$$\boldsymbol{S}_r^2 = \left(\boldsymbol{I} + 2\boldsymbol{\varepsilon}_{[0]}\right) \tag{3.25}$$

Hence, $\boldsymbol{S}_r$ can be replaced with $\boldsymbol{\varepsilon}_{[0]}$. Thus, dependence on $\boldsymbol{J}$ can be replaced with $\boldsymbol{\varepsilon}_{[0]}$. Likewise, using (3.24), we have

$$\boldsymbol{J}_{[i]} = \boldsymbol{R}(\boldsymbol{S}_r)_{[i]} + \boldsymbol{R}_{[i]}\boldsymbol{S}_r = (\boldsymbol{S}_l)_{[i]}\boldsymbol{R} + \boldsymbol{S}_l\boldsymbol{R}_{[i]};\ i = 1,2,\dots,n \tag{3.26}$$

and using (3.25)

$$(\boldsymbol{S}_r)_{[i]} = (\boldsymbol{S}_r)_{[i]}\big(\boldsymbol{\varepsilon}_{[k]};\ k = 0, 1, \ldots, m\big);\ i = 1, 2, \ldots, n \tag{3.27}$$

i.e., $(\boldsymbol{S}_r)_{[i]}$ can be expressed as a function of $\boldsymbol{\varepsilon}_{[k]};\ k = 0, 1, \ldots, m$. Thus, dependence on $\boldsymbol{J}_{[i]};\ i = 1, 2, \ldots, n$ can be replaced with $\boldsymbol{\varepsilon}_{[i]};\ i = 1, 2, \ldots, n$. With $\boldsymbol{\varepsilon}_{[0]}$ and $\boldsymbol{\varepsilon}_{[i]};\ i = 1, 2, \ldots, n$ as arguments, the conjugate stress measure must be $\boldsymbol{\sigma}^{[0]}$. The conclusion of replacing $\boldsymbol{\sigma}^*$ by $\boldsymbol{\sigma}^{[0]}$ can also be arrived at by using the relationship between $\boldsymbol{\sigma}^*$ and $\boldsymbol{\sigma}^{[0]}$ through $\boldsymbol{J}$ and then replacing $\boldsymbol{J}$ with $\boldsymbol{S}_r$ and then $\boldsymbol{S}_r$ by $\boldsymbol{\varepsilon}_{[0]}$. Hence, parallel to (3.21), we can write (using stress decomposition for $\boldsymbol{\sigma}^{[0]}$ similar to (3.12)):

$$\sigma_{jk}^{[0]} = {}_{e}\sigma_{jk}^{[0]} + {}_{d}\sigma_{jk}^{[0]}\big(\boldsymbol{\varepsilon}_{[0]}, \boldsymbol{\varepsilon}_{[i]};\ i = 1, 2, \ldots, n, \boldsymbol{g}, \theta\big) \tag{3.28}$$

$$\boldsymbol{q} = \boldsymbol{q}\big(\boldsymbol{\varepsilon}_{[0]}, \boldsymbol{\varepsilon}_{[i]};\ i = 1, 2, \ldots, n, \boldsymbol{g}, \theta\big) \tag{3.29}$$

$$\Phi = \Phi\big(\boldsymbol{J}, \theta\big) \tag{3.30}$$

and

$${}_{e}\boldsymbol{\sigma}^{*T} = \rho_0 \frac{\partial \Phi\big(\boldsymbol{J}, \theta\big)}{\partial \boldsymbol{J}} \tag{3.31}$$

In (3.30), we cannot change the dependence of Φ on $\boldsymbol{J}$ by $\boldsymbol{\varepsilon}_{[0]}$, due to (3.31). This is necessary and is also intentional, as this form is useful in the derivation of ${}_{e}\boldsymbol{\sigma}^{[0]}$ through ${}_{e}\boldsymbol{\sigma}^*$ defined by (3.31) using (3.30).

Equations (3.28) and (3.29) define the desired choices of dependent variables for the constitutive theories for $\boldsymbol{\sigma}^{[0]}$ and $\boldsymbol{q}$ and their argument tensors. $\boldsymbol{\varepsilon}_{[0]}$ and $\boldsymbol{\varepsilon}_{[i]};\ i = 1, 2, \ldots, n$ are symmetric tensors of rank two, $\boldsymbol{g}$ is a tensor of rank one, and θ is a tensor of rank zero.

3.4 Entropy inequality in Helmholtz free energy density Φ and conjugate pair $\boldsymbol{\sigma}^{[0]}$ and $\dot{\boldsymbol{\varepsilon}}$

Consider entropy inequality in Φ, $\boldsymbol{\sigma}^{[0]}$, and $\dot{\boldsymbol{\varepsilon}}$.

$$\rho_0(\dot{\Phi} + \eta\dot{\theta}) + \frac{|J|q_i g_i}{\theta} - \sigma_{ik}^{[0]}(\dot{\varepsilon})_{ik} \leq 0 \tag{3.32}$$

The preliminary choice of dependent variables in the constitutive theories can be made using (3.32), balance of momenta, and the first law of thermodynamics. The choice of Φ, $\boldsymbol{q}$, and η, the entropy density, is rather obvious from (3.32). Balance of momenta and the energy equation suggest $\boldsymbol{\sigma}^{[0]}$ as a dependent variable. This choice is also supported by (3.32). We also note that the choice of $\boldsymbol{q}$ as a dependent variable is also supported by the energy equation. Thus the preliminary choice of Φ, $\boldsymbol{q}$, η, and $\boldsymbol{\sigma}^{[0]}$ as dependent variables in the constitutive theories is in accordance with the axioms of constitutive theory that states self-observable quantities and those derived from these using direct differentiation and integration cannot be used as dependent variables in the constitutive theories. By the same reasoning, displacements $\boldsymbol{u}$, temperature θ, and temperature gradient $\boldsymbol{g}$ are ruled out as dependent variables in the constitutive theories. Since $\boldsymbol{\sigma}^{[0]}$ and $\dot{\boldsymbol{\varepsilon}}$ are conjugate, the choice of $\boldsymbol{\varepsilon}_{[0]}$ as an argument of the dependent variables is quite straightforward. The choice of θ and $\boldsymbol{g}$ as arguments is obvious too. The choice of $\dot{\boldsymbol{\varepsilon}}$ or $\boldsymbol{\varepsilon}_{[1]}$ as an argument tensor in addition to $\boldsymbol{\varepsilon}_{[0]}$, θ, and $\boldsymbol{g}$ is necessitated due to the dissipation mechanism in thermoviscoelastic solids without memory.

Generalization of $\boldsymbol{\varepsilon}$ and $\boldsymbol{\varepsilon}_{[1]}$ as arguments leads to consideration of $\boldsymbol{\varepsilon}_{[i]}$; $i = 0, 1, \ldots, n$, the material derivatives of the Green's strain tensor up to orders n, as arguments of the dependent variables in the constitutive theories in addition to θ and $\boldsymbol{g}$. Thus, now we have identified the dependent variables and their arguments of the constitutive theories for thermoviscoelastic solids without memory.

$$
\begin{aligned}
\Phi &= \Phi(\boldsymbol{\varepsilon}_{[0]}, \boldsymbol{\varepsilon}_{[i]};\; i = 1, 2, \ldots, n, \theta, \boldsymbol{g}) \\
\boldsymbol{\sigma}^{[0]} &= \boldsymbol{\sigma}^{[0]}(\boldsymbol{\varepsilon}_{[0]}, \boldsymbol{\varepsilon}_{[i]};\; i = 1, 2, \ldots, n, \theta, \boldsymbol{g}) \\
\eta &= \eta(\boldsymbol{\varepsilon}_{[0]}, \boldsymbol{\varepsilon}_{[i]};\; i = 1, 2, \ldots, n, \theta, \boldsymbol{g}) \\
\boldsymbol{q} &= \boldsymbol{q}(\boldsymbol{\varepsilon}_{[0]}, \boldsymbol{\varepsilon}_{[i]};\; i = 1, 2, \ldots, n, \theta, \boldsymbol{g})
\end{aligned}
\tag{3.33}
$$

Now since arguments of Φ are defined, we can obtain an explicit form of $\dot{\Phi}$ using $\Phi(\cdot)$ in (3.33).

$$
\dot{\Phi} = \frac{\partial \Phi}{\partial (\varepsilon_{[0]})_{kl}} (\dot{\varepsilon}_{[0]})_{kl} + \sum_{i=1}^{n} \frac{\partial \Phi}{\partial (\varepsilon_{[1]})_{kl}} (\dot{\varepsilon}_{[1]})_{kl} + \frac{\partial \Phi}{\partial \theta} \dot{\theta} + \frac{\partial \Phi}{\partial g_i} \dot{g}_i \tag{3.34}
$$

Substituting from (3.34) into (3.32) and collecting coefficients

$$
\begin{aligned}
&\left(\rho_0 \frac{\partial \Phi}{\partial (\varepsilon_{[0]})_{kl}} - \sigma_{lk}^{[0]} \right) (\dot{\varepsilon}_{[0]})_{kl} - \rho_0 \left(\frac{\partial \Phi}{\partial \theta} + \eta \right) \dot{\theta} \\
&\quad + \sum_{i=1}^{n} \rho_0 \frac{\partial \Phi}{\partial (\varepsilon_{[i]})_{kl}} (\dot{\varepsilon}_{[i]})_{kl} + \rho_0 \frac{\partial \Phi}{\partial g_i} \dot{g}_i + \frac{|J| q_i g_i}{\theta} \leq 0
\end{aligned}
\tag{3.35}
$$

In order for (3.35) to hold for arbitrary but admissible $\dot{\boldsymbol{\varepsilon}}_{[i]};\; i = 1, 2, \ldots, n$, $\dot{\boldsymbol{g}}$, and $\dot{\theta}$, the following must hold.

$$
\begin{aligned}
\left(\rho_0 \frac{\partial \Phi}{\partial (\varepsilon_{[0]})_{kl}} - \sigma_{lk}^{[0]} \right) (\dot{\varepsilon}_{[0]})_{kl} + \frac{|J| q_i g_i}{\theta} &\leq 0 \\
\rho_0 \left(\frac{\partial \Phi}{\partial \theta} + \eta \right) &= 0 \\
\rho_0 \frac{\partial \Phi}{\partial (\varepsilon_{[i]})_{kl}} &= 0;\; i = 1, 2, \ldots, n \\
\rho_0 \frac{\partial \Phi}{\partial g_i} &= 0;\; i = 1, 2, 3
\end{aligned}
\tag{3.36}
$$

Since ρ_0 is constant, ρ_0 in the last three equations in (3.36) can be dropped, and we have

$$\left(\rho_0 \frac{\partial \Phi}{\partial(\varepsilon_{[0]})_{kl}} - \sigma_{lk}^{[0]}\right)(\dot{\varepsilon}_{[0]})_{kl} + \frac{|J| q_i g_i}{\theta} \leq 0 \tag{3.37}$$

$$\frac{\partial \Phi}{\partial \theta} + \eta = 0 \tag{3.38}$$

$$\frac{\partial \Phi}{\partial(\varepsilon_{[i]})_{kl}} = 0;\ i = 1, 2, \ldots, n \tag{3.39}$$

$$\frac{\partial \Phi}{\partial g_i} = 0;\ i = 1, 2, 3 \tag{3.40}$$

Remarks. (1) Equation (3.39) implies that Φ is not a function of $\boldsymbol{\varepsilon}_{[i]}$; $i = 1, 2, \ldots, n$.

(2) Equation (3.40) implies that Φ is not a function of $\boldsymbol{g}$ either.

(3) From (3.38), we have $\eta = -\frac{\partial \Phi}{\partial \theta}$, hence η is deterministic from Φ, therefore η is not a dependent variable in the constitutive theories.

(4) The inequality in (3.37) is essential in the form in which it is stated. For example

$$\rho_0 \frac{\partial \Phi}{\partial(\varepsilon_{[0]})_{kl}} - \sigma_{lk}^{[0]} = 0 \text{ and } \frac{|J| q_i g_i}{\theta} \leq 0 \tag{3.41}$$

is inappropriate due to the fact that these imply that $\boldsymbol{\sigma}^{[0]}$ is not a function of $\boldsymbol{\varepsilon}_{[i]}$; $i = 1, 2, \ldots, n$ as Φ is not a function of $\boldsymbol{\varepsilon}_{[i]}$; $i = 1, 2, \ldots, n$, which is contrary to (3.34). We note that (3.37) in the stated form is unable to provide further details or mechanisms for deriving constitutive theories for $\boldsymbol{\sigma}^{[0]}$ and $\boldsymbol{q}$.

(5) Based on remarks 1-4, (3.34) reduces to

$$\begin{aligned} \Phi &= \Phi(\boldsymbol{\varepsilon}_{[0]}, \theta) \\ \boldsymbol{\sigma}^{[0]} &= \boldsymbol{\sigma}^{[0]}(\boldsymbol{\varepsilon}_{[0]}, \boldsymbol{\varepsilon}_{[i]};\ i = 1, 2, \ldots, n, \theta, \boldsymbol{g}) \\ \boldsymbol{q} &= \boldsymbol{q}(\boldsymbol{\varepsilon}_{[0]}, \boldsymbol{\varepsilon}_{[i]};\ i = 1, 2, \ldots, n, \theta, \boldsymbol{g}) \end{aligned} \tag{3.42}$$

Thus the constitutive theories for thermoviscoelastic solids without memory reduce to the determination of the dependence of $\boldsymbol{\sigma}^{[0]}$ and $\boldsymbol{q}$ on the deformation using (3.42).

3.4.1 Decomposition of $\sigma^{[0]}$

To proceed further using (3.37) resulting from the entropy inequality, we perform a decomposition of $\boldsymbol{\sigma}^{[0]}$ into equilibrium stress ${}_{e}\boldsymbol{\sigma}^{[0]}$ and deviatoric stress ${}_{d}\boldsymbol{\sigma}^{[0]}$.

$$\boldsymbol{\sigma}^{[0]} = {}_{e}\boldsymbol{\sigma}^{[0]} + {}_{d}\boldsymbol{\sigma}^{[0]} \tag{3.43}$$

in which we have the following

$${}_{e}\boldsymbol{\sigma}^{[0]} = {}_{e}\boldsymbol{\sigma}^{[0]}(\boldsymbol{\varepsilon}_{[0]}, \theta) \tag{3.44}$$

$${}_{d}\boldsymbol{\sigma}^{[0]} = {}_{d}\boldsymbol{\sigma}^{[0]}(\boldsymbol{\varepsilon}_{[0]}, \boldsymbol{\varepsilon}_{[i]};\, i = 1, 2, \ldots, n, \theta, \boldsymbol{g}) \tag{3.45}$$

That is, ${}_{e}\boldsymbol{\sigma}^{[0]}$ is not a function of $\boldsymbol{\varepsilon}_{[i]};\, i = 1, 2, \ldots, n$ and $\boldsymbol{g}$.

Substituting from (3.43) into (3.37), we obtain

$$\left(\rho_0 \frac{\partial \Phi}{\partial(\varepsilon_{[0]})_{ik}} - {}_{e}\sigma_{ki}^{[0]} - {}_{d}\sigma_{ki}^{[0]}\right)(\varepsilon_{[1]})_{ik} + \frac{|J| q_i g_i}{\theta} \leq 0 \tag{3.46}$$

or

$$\left(\rho_0 \frac{\partial \Phi}{\partial(\varepsilon_{[0]})_{ik}} - {}_{e}\sigma_{ki}^{[0]}\right)(\varepsilon_{[1]})_{ik} - {}_{d}\sigma_{ki}^{[0]}(\varepsilon_{[1]})_{ik} + \frac{|J| q_i g_i}{\theta} \leq 0 \tag{3.47}$$

Since Φ is not a function of $\boldsymbol{\varepsilon}_{[i]};\, i = 1, 2, \ldots, n$ (remark 1) and neither is ${}_{e}\boldsymbol{\sigma}^{[0]}$ (in (3.44)), ${}_{e}\boldsymbol{\sigma}^{[0]}$ must be derivable from

$${}_{e}\sigma_{ki}^{[0]} = \rho \frac{\partial \Phi}{\partial(\varepsilon_{[0]})_{ik}} \text{ or } ({}_{e}\boldsymbol{\sigma}^{[0]})^T = \rho_0 \frac{\partial \Phi}{\partial \boldsymbol{\varepsilon}_{[0]}} \tag{3.48}$$

Using (3.48), the inequality (3.47) reduces to

$$- {}_{d}\sigma_{ki}^{[0]}(\varepsilon_{[1]})_{ik} + \frac{|J| q_i g_i}{\theta} \leq 0 \tag{3.49}$$

If we assume

$$\frac{|J| q_i g_i}{\theta} \leq 0 \tag{3.50}$$

then (3.49) is satisfied if

$${}_{d}\sigma^{[0]}_{ki} (\varepsilon_{[1]})_{ik} > 0 \tag{3.51}$$

Equation (3.51) requires that the work expended due to deviatoric stress ${}_{d}\boldsymbol{\sigma}^{[0]}$ be positive. In view of (3.48), the stress decomposition (3.43) can be written as

$$\sigma^{[0]}_{ij} = \rho_0 \frac{\partial \Phi}{\partial (\varepsilon_{[0]})_{ji}} + {}_{d}\sigma^{[0]}_{ij} (\boldsymbol{\varepsilon}_{[0]}, \boldsymbol{\varepsilon}_{[i]};\, i = 1, 2, \dots, n, \theta, \boldsymbol{g}) \tag{3.52}$$

Additionally, we have

$$\boldsymbol{q} = \boldsymbol{q}(\boldsymbol{\varepsilon}_{[0]}, \boldsymbol{\varepsilon}_{[i]};\, i = 1, 2, \dots, n, \theta, g) \tag{3.53}$$

$$\Phi = \Phi(\boldsymbol{\varepsilon}_{[0]}, \theta) \tag{3.54}$$

and

$$({}_{e}\boldsymbol{\sigma}^{[0]})^T = \rho_0 \frac{\partial \Phi}{\partial \boldsymbol{\varepsilon}_{[0]}} \tag{3.55}$$

Remarks. (1) We note that (3.52) – (3.55) derived using $\boldsymbol{\sigma}^{[0]}$ and $\dot{\boldsymbol{\varepsilon}}_{[0]}$ are exactly the same as those derived using conjugate pair $\boldsymbol{\sigma}^*$ and $\dot{\boldsymbol{J}}$, *i.e.*, (3.28) – (3.31). Equations (3.30)and (3.31) may appear different from (3.54) and (3.55), but they are the same, as $\boldsymbol{\varepsilon}_{[0]}$ and $\boldsymbol{J}$ and $\boldsymbol{\sigma}^*$ and $\boldsymbol{\sigma}^{[0]}$ are related (see reference [1] for details).

(2) When $\boldsymbol{\varepsilon}_{[i]};\, i = 1, 2, \dots, n$ (all or any) are arguments of the dependent variables in the constitutive theories, then

(a) Entropy inequality requires decomposition of the stress tensor into equilibrium and deviatoric stress tensors.

(b) Equilibrium stress ${}_{e}\boldsymbol{\sigma}^*$ or ${}_{e}\boldsymbol{\sigma}^{[0]}$ is deterministic from (3.17) or (3.55), but the deviatoric stress is not. We shall consider theory of generators and invariants [1–3] for deriving constitutive theories for ${}_{d}\boldsymbol{\sigma}^{[0]}$ and $\boldsymbol{q}$.

3.5 Constitutive theory for equilibrium stress ${}_{e}\boldsymbol{\sigma}^{[0]}$

As is well known, the need for stress tensor decomposition resulting in equilibrium and deviatoric stress tensors arises in fluids. When considering non-rate, or zero-rate constitutive theories in Lagrangian description using the conjugate pair $\boldsymbol{\sigma}^{[0]}$ and $\boldsymbol{\varepsilon}_{[0]}$, there is no need for decomposition of $\boldsymbol{\sigma}^{[0]}$ into equilibrium and deviatoric tensors. However, as shown here, when deriving rate constitutive theories in Lagrangian description, the decomposition of $\boldsymbol{\sigma}^{[0]}$ into equilibrium and deviatoric stress tensors is necessary. In most published works for fluids, the Cauchy stress decomposition into equilibrium and deviatoric tensors is substituted in the entropy inequality to arrive at conditions that enable derivation of the constitutive theory for the equilibrium Cauchy stress tensor. The physical meanings of the equilibrium Cauchy stress tensor as thermodynamic and mechanical pressure for compressible and incompressible matter are established only after the derivation of the constitutive theory for the equilibrium stress tensor. This approach is contrary to the basic philosophy of continuum mechanics, in which we consider some desired physics (often limited physics of interest) for which we derive the mathematical description using principles and axioms of continuum mechanics.

The objectives of the material that follows are: (i) first, to present definitions and describe physics represented by the equilibrium and deviatoric stress tensors (ii) then present a derivation of the equilibrium Cauchy stress tensor (contravariant case considered here) using the physics described by it and the conditions resulting from the entropy inequality (iii) finally, use the derivation in (ii) to present constitutive theory for ${}_{e}\boldsymbol{\sigma}^{[0]}$. In this approach, we address physics first, followed by the mathematical details of the derivations based on entropy inequality, and then show that this derivation supports the physics.

For constitutive theories, it is most meaningful to consider the deformed matter in the current configuration at a material point $\bar{P}(\bar{x}_i)$. Let $\bar{\boldsymbol{\sigma}}^{(0)}$, the contravariant Cauchy stress tensor be the measure of stress in the current configuration. Based on decomposition of $\bar{\boldsymbol{\sigma}}^{(0)}$ into equilibrium stress ${}_e\bar{\boldsymbol{\sigma}}^{(0)}$ and deviatoric stress ${}_d\bar{\boldsymbol{\sigma}}^{(0)}$, we have

$$\bar{\boldsymbol{\sigma}}^{(0)} = {}_e\bar{\boldsymbol{\sigma}}^{(0)} + {}_d\bar{\boldsymbol{\sigma}}^{(0)} \tag{3.56}$$

or

$${}_d\bar{\boldsymbol{\sigma}}^{(0)} = \bar{\boldsymbol{\sigma}}^{(0)} - {}_e\bar{\boldsymbol{\sigma}}^{(0)} \tag{3.57}$$

Thus, the deviatoric stress tensor is obtained by subtracting the equilibrium stress tensor from the total Cauchy stress tensor $\bar{\boldsymbol{\sigma}}^{(0)}$.

Consider compressible matter. By definition, the equilibrium stress due to deformation field results in a change of volume of the matter but no distortion of the volume, while ${}_d\bar{\boldsymbol{\sigma}}^{(0)}$ due to the deformation field causes only a change in shape of the volume without any change in the volume of the matter. This holds regardless of whether the matter is solid or fluid and is irrespective of the magnitude of deformation. Thus, ${}_e\bar{\boldsymbol{\sigma}}^{(0)}$ must be a diagonal tensor in which the diagonal elements are of the same magnitude, *i.e.*, we can represent ${}_e\bar{\boldsymbol{\sigma}}^{(0)}$ by

$${}_e\bar{\boldsymbol{\sigma}}^{(0)} = \bar{p}\boldsymbol{I} \tag{3.58}$$

We note that ${}_e\bar{\boldsymbol{\sigma}}^{(0)}$ in (3.58) is frame invariant, thus ${}_e\bar{\boldsymbol{\sigma}}^{(0)}$ is like a pressure field. The arguments of $\bar{p}$ can be easily determined. Consider a unit cube with ${}_e\bar{\boldsymbol{\sigma}}^{(0)}$ acting on its faces. A compressive ${}_e\bar{\boldsymbol{\sigma}}^{(0)}$ will result in uniform shortening of the size of the cube (without change in shape), *i.e.*, change in density of the volume of matter in the cube. Likewise, tensile ${}_e\bar{\boldsymbol{\sigma}}^{(0)}$ will uniformly enlarge the size of the cube, (again without change in shape), resulting in decrease of the density of the volume of matter in the cube. If the motion of the faces of the cube is constrained and if the cube is subjected to a uniform decrease in temperature θ, the faces of the cube will experience positive (*i.e.*, tensile) ${}_e\bar{\boldsymbol{\sigma}}^{(0)}$. On the other hand, if the cube is subjected to a uniform increase in temperature, the faces of the cube will experience compressive (*i.e.*, negative) ${}_e\bar{\sigma}^{(0)}$. Thus, we conclude that for compressible matter, $\bar{p}$ must be a function of density $\bar{\rho}(\bar{x}_i, t)$ and temperature $\bar{\theta}(\bar{x}_i, t)$ in the current configuration.

Since from the conservation of mass (continuity equation), we have

$$\begin{aligned} \rho_0 &= \rho(x_i, t)|J| \\ \text{or } \rho_0 &= \bar{\rho}(\bar{x}_i, t)|J| \end{aligned} \tag{3.59}$$

the dependence of $\bar{p}$ on $\bar{\rho}$ can be replaced by dependence on $|J|$. It is a matter of convenience whether we consider $\bar{p}$ or ${}_e\bar{\boldsymbol{\sigma}}^{(0)}$ as a function of $(|J|, \theta)$, $(\frac{1}{\bar{\rho}}, \bar{\theta})$, or simply $(\bar{\rho}, \bar{\theta})$, all three choices of arguments of $\bar{p}$ are equivalent.

If we consider incompressible matter, then there is no change in volume, *i.e.*, ${}_e\bar{\boldsymbol{\sigma}}^{(0)}$ acting on a unit cube causes no change in the volume of the cube. Hence, for this case, (3.58) holds with $\bar{\rho} = \rho_0$ and $|J| = 1$, which implies that $\bar{p} = \bar{p}(\bar{\theta})$ for the incompressible case.

From entropy inequality, we note that (3.31) must hold. In the following, we use (3.31) and the fact that ${}_e\bar{\boldsymbol{\sigma}}^{(0)}$ is dependent on $\bar{\rho}$ and $\bar{\theta}$ for compressible matter to derive the explicit form of $\bar{p}$ as a function of Helmholtz free energy density. In the case of incompressible matter we also use condition (3.31) or entropy inequality along with the incompressibility constraint to derive the constitutive theory for ${}_e\bar{\boldsymbol{\sigma}}^{(0)}$.

3.5.1 Compressible matter: equilibrium stress tensor ${}_e\bar{\boldsymbol{\sigma}}^{(0)}$

Consider the condition (3.31) resulting from the entropy inequality

$${}_e\boldsymbol{\sigma}^{*T} = \rho_0 \frac{\partial \Phi(\boldsymbol{J}, \theta)}{\partial \boldsymbol{J}} \tag{3.60}$$

For compressible matter

$${}_e\bar{\boldsymbol{\sigma}}^{(0)} = |J|^{-1}\, {}_e\boldsymbol{\sigma}^{*T} \boldsymbol{J}^T \tag{3.61}$$

${}_{e}\boldsymbol{\sigma}^{*}$ is in Lagrangian description, whereas ${}_{e}\bar{\boldsymbol{\sigma}}^{(0)}$ is in Eulerian description. It is clear that $\boldsymbol{J}$ must be replaced by $|J|$, as equilibrium stress is a function of $|J|$. But, due to continuity, $|J| = \frac{\rho_0}{\bar{\rho}}$, thus $|J|$ can be replaced by $\frac{1}{\bar{\rho}}$ or simply $\bar{\rho}$ or $\bar{v}$ where $\bar{v}$ is specific volume (equal to $\frac{1}{\bar{\rho}}$). In other words, in Eulerian description, we can consider

$$\bar{\Phi} = \bar{\Phi}(|J|, \theta) \text{ or } \bar{\Phi} = \bar{\Phi}(\bar{\rho}, \bar{\theta}) \text{ or } \bar{\Phi} = \bar{\Phi}(\frac{1}{\bar{\rho}}, \bar{\theta}) \text{ or } \bar{\Phi} = \bar{\Phi}(\bar{v}, \bar{\theta}) \tag{3.62}$$

If we consider $\bar{\Phi} = \bar{\Phi}(\bar{\rho}, \bar{\theta})$, then (3.60) and (3.61) yield

$${}_{e}\bar{\boldsymbol{\sigma}}^{(0)} = |J|^{-1} \rho_0 \frac{\partial \bar{\Phi}(\bar{\rho}, \bar{\theta})}{\partial \boldsymbol{J}} \boldsymbol{J}^T \tag{3.63}$$

Using $\bar{\rho} = \bar{\rho}(|J|)$ and the chain rule of differentiation, we can write

$$\frac{\partial \bar{\Phi}(\bar{\rho}, \bar{\theta})}{\partial \boldsymbol{J}} = \frac{\partial \bar{\Phi}(\bar{\rho}, \bar{\theta})}{\partial \bar{\rho}} \frac{\partial \bar{\rho}}{\partial |J|} \frac{\partial |J|}{\partial \boldsymbol{J}} \tag{3.64}$$

From conservation of mass

$$\rho_0 = \bar{\rho}|J| \text{ or } \bar{\rho} = \frac{\rho_0}{|J|} \tag{3.65}$$

$$\therefore \frac{\partial \bar{\rho}}{\partial |J|} = -\frac{\rho_0}{|J|^2} = -\frac{\bar{\rho}^2}{\rho_0} \tag{3.66}$$

and

$$\frac{\partial |J|}{\partial \boldsymbol{J}} = (\boldsymbol{J}^{-1})^T |J| \tag{3.67}$$

Substituting from (3.66) and (3.67) into (3.64)

$$\frac{\partial \bar{\Phi}(\bar{\rho}, \bar{\theta})}{\partial \boldsymbol{J}} = \frac{\partial \bar{\Phi}(\bar{\rho}, \bar{\theta})}{\partial \bar{\rho}} \left(-\frac{\bar{\rho}^2}{\rho_0} \right) (\boldsymbol{J}^{-1})^T |J| \tag{3.68}$$

Substituting from (3.68) into (3.63)

$$ {}_{e}\bar{\boldsymbol{\sigma}}^{(0)} = |J|^{-1}\rho_0 \left(\frac{\partial\bar{\Phi}(\bar{\rho},\bar{\theta})}{\partial\bar{\rho}} \left(-\frac{\bar{\rho}^2}{\rho_0}\right) (\boldsymbol{J}^{-1})^T \right) \boldsymbol{J}^T|J| \tag{3.69} $$

or

$$ {}_{e}\bar{\boldsymbol{\sigma}}^{(0)} = -\bar{\rho}^2 \frac{\partial\bar{\Phi}(\bar{\rho},\bar{\theta})}{\partial\bar{\rho}} (\boldsymbol{J}^{-1})^T \boldsymbol{J}^T \tag{3.70} $$

Since $(\boldsymbol{J}^{-1})^T\boldsymbol{J}^T = \left(\boldsymbol{J}\boldsymbol{J}^{-1}\right)^T = \boldsymbol{I}^T = \boldsymbol{I}$, we can write the following for ${}_{e}\bar{\boldsymbol{\sigma}}^{(0)}$

$$ {}_{e}\bar{\boldsymbol{\sigma}}^{(0)} = -\bar{\rho}^2 \frac{\partial\bar{\Phi}(\bar{\rho},\bar{\theta})}{\partial\bar{\rho}} \boldsymbol{I} \tag{3.71} $$

If we let

$$ \bar{p}(\bar{\rho},\bar{\theta}) = -\bar{\rho}^2 \frac{\partial\bar{\Phi}(\bar{\rho},\bar{\theta})}{\partial\bar{\rho}} \tag{3.72} $$

then

$$ {}_{e}\bar{\boldsymbol{\sigma}}^{(0)} = \bar{p}(\bar{\rho},\bar{\theta})\boldsymbol{I} \tag{3.73} $$

Equation (3.72) defines $\bar{p}(\bar{\rho},\bar{\theta})$. Knowing $\bar{\Phi}(\bar{\rho},\bar{\theta})$, $\bar{p}(\bar{\rho},\bar{\theta})$ is strictly deterministic from (3.72). $\bar{p}(\bar{\rho},\bar{\theta})$ is the equation of state. This theory also admits $\bar{p}(\bar{\rho},\bar{\theta})$ derived empirically, experimentally, or using any other approach such as kinetic theory. $\bar{p}(\bar{\rho},\bar{\theta})$ is called thermodynamic pressure. If we assume compressive pressure to be positive, then $\bar{p}(\bar{\rho},\bar{\theta})$ in (3.73) can be replaced by $-\bar{p}(\bar{\rho},\bar{\theta})$.

Remarks. The constitutive theory (3.73) for ${}_{e}\bar{\boldsymbol{\sigma}}^{(0)}$ can also be derived assuming $\bar{\Phi} = \bar{\Phi}(\bar{v},\bar{\theta})$. In this case,

$$ \bar{p}(\bar{v},\bar{\theta}) = \frac{\partial\bar{\Phi}(\bar{v},\bar{\theta})}{\partial\bar{v}} \tag{3.74} $$

By using $\bar{\rho} = \frac{1}{\bar{v}}$ or $\bar{v} = \frac{1}{\bar{\rho}}$, it is straightforward to show that the definitions of $\bar{p}(\cdot,\cdot)$ in (3.72) and (3.74) are precisely equivalent.

3.5.2 Incompressible matter: equilibrium stress tensor ${}_e\bar{\sigma}^{(0)}$

Recall the condition resulting from the entropy inequality in Lagrangian description using conjugate pair $\boldsymbol{\sigma}^*, \dot{\boldsymbol{J}}$ after the substitution of stress decomposition $\boldsymbol{\sigma}^* = {}_e\boldsymbol{\sigma}^* + {}_d\boldsymbol{\sigma}^*$, i.e. (3.31) after replacing $\boldsymbol{J}$ with $|J|$ and then with $\rho(x,t)$ yields

$$
{}_e\boldsymbol{\sigma}^{*T} = \rho_0 \frac{\partial \Phi\big(\rho(x,t), \theta(x,t)\big)}{\partial \boldsymbol{J}(x,t)} \tag{3.75}
$$

For incompressible matter, $\rho(x,t) = \rho_0(x) = \text{Constant}$, which implies that $|J| = 1$, hence $\Phi = \Phi(\theta)$ and $\frac{\partial \Phi}{\partial \boldsymbol{J}} = 0$, thus, ${}_e\bar{\sigma}^{(0)}$ cannot be derived using a derivation similar to the compressible case presented in section 3.5.1. Instead, the incompressibility condition $|J| = 1$ must be enforced. For incompressible matter:

$$
\operatorname{tr}(\bar{\boldsymbol{D}}) = \operatorname{tr}(\bar{\boldsymbol{L}}) = \operatorname{tr}\left(\dot{\boldsymbol{J}}\boldsymbol{J}^{-1}\right) = \dot{J}_{ik}(J^{-1})_{ki} = 0 \tag{3.76}
$$

We enforce (3.76) through the entropy inequality. If (3.76) holds, then

$$
p\dot{J}_{ik}(J^{-1})_{ki} = p(\theta)\dot{J}_{ik}(J^{-1})_{ki} = 0 \tag{3.77}
$$

must also hold, where p is a Lagrange multiplier. p cannot be a function of the Jacobian but can depend on temperature θ, *i.e.*, $p(\theta)$ is valid. We add (3.77) to the left side of the entropy inequality (3.1). Since (3.77) is zero, it does not change the meaning of the entropy inequality:

$$
\left(\rho_0 \frac{\partial \Phi}{\partial J_{ik}} - {}_e\sigma^*_{ki}\right)\dot{J}_{ki} - {}_d\sigma^*_{ki}\dot{J}_{ik} + \frac{|J| q_i g_i}{\theta} + p(\theta)\dot{J}_{ik}(J^{-1})_{ki} \le 0 \tag{3.78}
$$

Regrouping the terms and using $\rho_0 \frac{\partial \Phi}{\partial J_{ik}} = 0$ in (3.78)

$$
\big(p(\theta)(J^{-1})_{ki} - {}_e\sigma^*_{ik}\big)\dot{J}_{ik} - {}_d\sigma^*_{ki}\dot{J}_{ik} + \frac{|J| q_i g_i}{\theta} \le 0 \tag{3.79}
$$

Inequality (3.79) holds if

$$p(\theta)(J^{-1})_{ki} - {}_{e}\sigma^{*}_{ik} = 0 \tag{3.80}$$

$${}_{d}\sigma^{*}_{ki}\dot{J}_{ik} > 0 \tag{3.81}$$

and

$$q_i g_i \leq 0; \text{ (as } |J| = 1 \text{ and } \theta > 0) \tag{3.82}$$

From (3.80)

$${}_{e}\boldsymbol{\sigma}^{*T} = p(\theta)(\boldsymbol{J}^{T})^{-1} \tag{3.83}$$

For incompressible matter

$$\bar{\boldsymbol{\sigma}}^{(0)} = \boldsymbol{\sigma}^{*T}\boldsymbol{J}^{T} \tag{3.84}$$

Hence, using (3.80) and (3.84), we have

$${}_{e}\bar{\boldsymbol{\sigma}}^{(0)} = \bar{p}(\bar{\theta})\boldsymbol{I} \tag{3.85}$$

$\bar{p}(\bar{\theta})$ is called mechanical pressure. $\bar{p}(\bar{\theta})$ is not deterministic from the deformation field as it is an arbitrary Lagrange multiplier but can depend upon temperature $\bar{\theta}$. If we define compressive mechanical pressure to be positive, then $\bar{p}(\bar{\theta})$ in (3.85) can be replaced by $-\bar{p}(\bar{\theta})$. We note that ${}_{e}\bar{\sigma}^{(0)}$ for the incompressible case is also frame invariant; a rigid rotation of the current configuration has no effect on it.

3.5.3 Equilibrium stress ${}_{e}\boldsymbol{\sigma}^{[0]}$

First, we note that the Lagrangian description of contravariant equilibrium Cauchy stress tensors for compressible and incompressible matter can be written as (using (3.61) and

(3.85))

$$ {}_e\boldsymbol{\sigma}^{(0)} = p(\rho,\theta)\boldsymbol{I};\ \text{compressible} \tag{3.86} $$

$$ {}_e\boldsymbol{\sigma}^{(0)} = p(\theta)\boldsymbol{I};\ \text{incompressible} \tag{3.87} $$

Recall that [1]

$$ \boldsymbol{\sigma}^{[0]} = |\boldsymbol{J}|\boldsymbol{J}^{-1}\bar{\boldsymbol{\sigma}}^{(0)}(\boldsymbol{J}^{-1})^T;\ \text{compressible} \tag{3.88} $$

$$ \boldsymbol{\sigma}^{[0]} = \boldsymbol{J}^{-1}\bar{\boldsymbol{\sigma}}^{(0)}(\boldsymbol{J}^{-1})^T;\ \text{incompressible} \tag{3.89} $$

Using (3.86) and (3.87) in (3.88) and (3.89), we can obtain

$$ {}_e\boldsymbol{\sigma}^{[0]} = p(\rho,\theta)|\boldsymbol{J}|(\boldsymbol{J}^T\boldsymbol{J})^{-1};\ \text{compressible} \tag{3.90} $$

$$ {}_e\boldsymbol{\sigma}^{[0]} = p(\theta)(\boldsymbol{J}^T\boldsymbol{J})^{-1};\ \text{incompressible} \tag{3.91} $$

Remarks. (1) Unlike ${}_e\bar{\boldsymbol{\sigma}}^{(0)}$ or ${}_e\boldsymbol{\sigma}^{(0)}$, ${}_e\boldsymbol{\sigma}^{[0]}$ is not a pressure field, *i.e.*, ${}_e\boldsymbol{\sigma}^{[0]}$ is not a diagonal tensor. This is fine as ${}_e\boldsymbol{\sigma}^{[0]}$ is a hypothetical equilibrium stress acting on an undeformed tetrahedron in the reference configuration. The definitions of ${}_e\boldsymbol{\sigma}^{[0]}$ in (3.90) and (3.91) will produce the pressure field ${}_e\bar{\boldsymbol{\sigma}}^{(0)}$ or ${}_e\boldsymbol{\sigma}^{(0)}$ corresponding to the deformed tetrahedron in the current configuration.

(2) In the case of infinitesimal deformation ($x \approx \bar{x}$), ${}_e\boldsymbol{\sigma}^{[0]}$ in (3.90) and (3.91) are diagonal tensors.

3.6 Constitutive theory for deviatoric second Piola-Kirchhoff stress tensor ${}_d\boldsymbol{\sigma}^{[0]}$ and heat vector $\boldsymbol{q}$

In view of the derivation of the equilibrium stress, the derivation of the constitutive theory now reduces to the constitutive theory for ${}_d\boldsymbol{\sigma}^{[0]}$ and $\boldsymbol{q}$ using (3.28) and(3.29) in which the constitutive theory for ${}_{e}\boldsymbol{\sigma}^{[0]}$ is now known, hence we have

$$\sigma_{jk}^{[0]} = p|J|(J_{kj}J_{jk})^{-1} + {}_{d}\sigma_{jk}^{[0]}\big(\boldsymbol{\varepsilon}_{[0]}, \boldsymbol{\varepsilon}_{[i]};\ i = 1, 2, \dots, n, \boldsymbol{g}, \theta\big) \tag{3.92}$$

$$\boldsymbol{q} = \boldsymbol{q}\big(\boldsymbol{\varepsilon}_{[0]}, \boldsymbol{\varepsilon}_{[i]};\ i = 1, 2, \dots, n, \boldsymbol{g}, \theta\big) \tag{3.93}$$

in which $p = p\big(\rho(\boldsymbol{x}, t), \theta(\boldsymbol{x}, t)\big)$ for compressible matter and $p = p\big(\theta(\boldsymbol{x}, t)\big)$ and $|J| = 1$ for the incompressible case.

We derive constitutive theories for ${}_{d}\boldsymbol{\sigma}^{[0]}$ and $\boldsymbol{q}$ using the theory of generators and invariants [1–3, 31]. Details are given in the following.

3.6.1 Rate constitutive theory of order n for ${}_{d}\boldsymbol{\sigma}^{[0]}$

In this theory we consider the most general case in which the argument tensors of ${}_{d}\boldsymbol{\sigma}^{[0]}$ and $\boldsymbol{q}$ are given by (3.92) and (3.93). In the theory of generators and invariants, we express ${}_{d}\boldsymbol{\sigma}^{[0]}$ as a linear combination of $\boldsymbol{I}$ and the combined generators of the argument tensors of ${}_{d}\boldsymbol{\sigma}^{[0]}$. The coefficients in the linear combinations are functions of the combined invariants of the argument tensors of ${}_{d}\boldsymbol{\sigma}^{[0]}$. The material coefficients are derived by considering Taylor series expansion of each coefficient in the linear combination in the combined invariants and temperature θ. ${}_{d}\boldsymbol{\sigma}^{[0]}$ is a symmetric tensor of rank two. The argument tensors $\boldsymbol{\varepsilon}_{[0]}$ and $\boldsymbol{\varepsilon}_{[i]};\ i = 1, 2, \dots, n$ are also symmetric tensors of rank two, but $\boldsymbol{g}$ and θ are tensors of rank one and zero. A complete list of generators and invariants appears in Appendix A. Let ${}^{\sigma}\underset{\sim}{\boldsymbol{G}}^{i};\ i = 1, 2, \dots, N$ be the combined generators [31] that are tensors of rank two and let ${}^{q\sigma}\underset{\sim}{I}^{j};\ j = 1, 2, \dots, M$ be the combined invariants of the same argument tensors. Then we can express ${}_{d}\boldsymbol{\sigma}^{[0]}$ in the current configuration using

$${}_{d}\boldsymbol{\sigma}^{[0]} = {}^{\sigma}\underset{\sim}{\alpha}^{0}\boldsymbol{I} + \sum_{i=1}^{n} {}^{\sigma}\underset{\sim}{\alpha}^{i}\,{}^{\sigma}\underset{\sim}{\boldsymbol{G}}^{i} \tag{3.94}$$

in which

$${}^{\sigma}\underset{\sim}{\alpha}^{i} = {}^{\sigma}\underset{\sim}{\alpha}^{i}\big({}^{q\sigma}\underset{\sim}{I}^{j};\ j = 1, 2, \dots, M, \theta\big);\ i = 0, 1, \dots, N \tag{3.95}$$

To determine material coefficients from ${}^{\sigma}\underset{\sim}{\alpha}^i;\ i=0,1,\dots,N$ in (3.94), we consider Taylor series expansion of the coefficients ${}^{\sigma}\underset{\sim}{\alpha}^i;\ i=0,1,\dots,N$ about a known configuration $\underline{\Omega}$ in ${}^{q\sigma}\underset{\sim}{I}^j;\ j=1,2,\dots,M$ and θ and retain only up to linear terms in the invariants and temperature θ.

$$
{}^{\sigma}\underset{\sim}{\alpha}^i = {}^{\sigma}\underset{\sim}{\alpha}^i|_{\underline{\Omega}} + \sum_{j=1}^{M} \left.\frac{\partial\, {}^{\sigma}\underset{\sim}{\alpha}^i}{\partial\, {}^{q\sigma}\underset{\sim}{I}^j}\right|_{\underline{\Omega}} \left({}^{q\sigma}\underset{\sim}{I}^j - ({}^{q\sigma}\underset{\sim}{I}^j)_{\underline{\Omega}}\right) + \left.\frac{\partial\, {}^{\sigma}\underset{\sim}{\alpha}^i}{\partial\theta}\right|_{\underline{\Omega}} (\theta-\theta_{\underline{\Omega}});\ i=0,1,\dots,N \tag{3.96}
$$

When (3.96) is substituted in (3.94), we obtain the final form of the most general rate constitutive theory for ${}_{d}\boldsymbol{\sigma}^{[0]}$ of order n. Details of the resulting material coefficients are given in the following.

Substituting from (3.96) into (3.94)

$$
\begin{aligned}
{}_{d}\boldsymbol{\sigma}^{[0]} = &\left({}^{\sigma}\underset{\sim}{\alpha}^0|_{\underline{\Omega}} + \sum_{j=1}^{M} \left.\frac{\partial\, {}^{\sigma}\underset{\sim}{\alpha}^0}{\partial\, {}^{q\sigma}\underset{\sim}{I}^j}\right|_{\underline{\Omega}} \left({}^{q\sigma}\underset{\sim}{I}^j - ({}^{q\sigma}\underset{\sim}{I}^j)_{\underline{\Omega}}\right) + \left.\frac{\partial\, {}^{\sigma}\underset{\sim}{\alpha}^0}{\partial\theta}\right|_{\underline{\Omega}} (\theta-\theta_{\underline{\Omega}}) \right)\boldsymbol{I} \\
&+ \sum_{i=1}^{N}\left({}^{\sigma}\underset{\sim}{\alpha}^i|_{\underline{\Omega}} + \sum_{j=1}^{M} \left.\frac{\partial\, {}^{\sigma}\underset{\sim}{\alpha}^i}{\partial\, {}^{q\sigma}\underset{\sim}{I}^j}\right|_{\underline{\Omega}} \left({}^{q\sigma}\underset{\sim}{I}^j - ({}^{q\sigma}\underset{\sim}{I}^j)_{\underline{\Omega}}\right) + \left.\frac{\partial\, {}^{\sigma}\underset{\sim}{\alpha}^i}{\partial\theta}\right|_{\underline{\Omega}} (\theta-\theta_{\underline{\Omega}}) \right){}^{\sigma}\underset{\sim}{\boldsymbol{G}}^i
\end{aligned} \tag{3.97}
$$

Collecting coefficients (those defined in known configuration $\underline{\Omega}$) of $\boldsymbol{I}$, ${}^{q\sigma}\underset{\sim}{I}^j\boldsymbol{I}$, ${}^{\sigma}\underset{\sim}{\boldsymbol{G}}^i$, ${}^{q\sigma}\underset{\sim}{I}^j\,{}^{\sigma}\underset{\sim}{\boldsymbol{G}}^i$, $(\theta-\theta_{\underline{\Omega}}){}^{\sigma}\underset{\sim}{\boldsymbol{G}}^i$, and $(\theta-\theta_{\underline{\Omega}})\boldsymbol{I}$ in (3.97) and defining

$$
\begin{aligned}
&\underline{\boldsymbol{\sigma}}^0|_{\underline{\Omega}} = {}^{\sigma}\underset{\sim}{\alpha}^0|_{\underline{\Omega}} + \sum_{j=1}^{M} \left.\frac{\partial\, {}^{\sigma}\underset{\sim}{\alpha}^0}{\partial\, {}^{q\sigma}\underset{\sim}{I}^j}\right|_{\underline{\Omega}} ({}^{q\sigma}\underset{\sim}{I}^j)_{\underline{\Omega}} && {}^{\sigma}\underline{a}_j = \left.\frac{\partial\, {}^{\sigma}\underset{\sim}{\alpha}^0}{\partial\, {}^{q\sigma}\underset{\sim}{I}^j}\right|_{\underline{\Omega}};\ j=1,2,\dots,M \\
&{}^{\sigma}\underline{b}_i = {}^{\sigma}\underset{\sim}{\alpha}^i|_{\underline{\Omega}} + \sum_{j=1}^{M} \left.\frac{\partial\, {}^{\sigma}\underset{\sim}{\alpha}^i}{\partial\, {}^{q\sigma}\underset{\sim}{I}^j}\right|_{\underline{\Omega}};\ i=1,2,\dots,N && {}^{\sigma}\underline{c}_{ij} = \left.\frac{\partial\, {}^{\sigma}\underset{\sim}{\alpha}^i}{\partial\, {}^{q\sigma}\underset{\sim}{I}^j}\right|_{\underline{\Omega}};\ \begin{matrix} i=1,2,\dots,N \\ j=1,2,\dots,M \end{matrix} \\
&{}^{\sigma}\underline{d}_i = -\left.\frac{\partial\, {}^{\sigma}\underset{\sim}{\alpha}^i}{\partial\theta}\right|_{\underline{\Omega}};\ i=1,2,\dots,N && (\underline{\alpha}_{tm})_{\underline{\Omega}} = -\left.\frac{\partial\, {}^{\sigma}\underset{\sim}{\alpha}^0}{\partial\theta}\right|_{\underline{\Omega}}
\end{aligned} \tag{3.98}
$$

Equation (3.97) can be written as

$$
\begin{aligned}
{}_d\boldsymbol{\sigma}^{[0]} = {} & \underline{\boldsymbol{\sigma}}^0_{\underline{\Omega}}\boldsymbol{I} + \sum_{j=1}^{M} {}^{\sigma}\underline{a}_j\, {}^{q\sigma}\underset{\sim}{I}^j \boldsymbol{I} + \sum_{i=1}^{N} {}^{\sigma}\underline{b}_i {}^{\sigma}\underset{\sim}{\boldsymbol{G}}^i \\
& + \sum_{i=1}^{N}\sum_{j=1}^{M} {}^{\sigma}\underline{c}_{ij}\, {}^{q\sigma}\underset{\sim}{I}^j {}^{\sigma}\underset{\sim}{\boldsymbol{G}}^i + \sum_{i=1}^{N} {}^{\sigma}\underline{d}_i(\theta - \theta_{\underline{\Omega}}) {}^{\sigma}\underset{\sim}{\boldsymbol{G}}^i \\
& - (\underline{\alpha}_{tm})_{\underline{\Omega}}(\theta - \theta_{\underline{\Omega}})\boldsymbol{I}
\end{aligned}
\tag{3.99}
$$

$\underline{\boldsymbol{\sigma}}^0|_{\underline{\Omega}}$ is the initial stress in the configuration $\underline{\Omega}$. This constitutive theory for ${}_d\boldsymbol{\sigma}^{[0]}$ requires $(M + N + MN + N + 1)$ material coefficients. The material coefficients defined in (3.98) are functions of $\theta_{\underline{\Omega}}$ and $({}^{q\sigma}\underset{\sim}{I}^j)_{\underline{\Omega}};\ j = 1, 2, \dots, M$ in the known configuration $\underline{\Omega}$. This constitutive theory is based on integrity and hence is complete. Tables 1 and 2 give combined generators ${}^{\sigma}\underset{\sim}{\boldsymbol{G}}^i;\ i = 1, 2, \dots, 12$ and combined invariants ${}^{q\sigma}\underset{\sim}{I}^j;\ j = 1, 2, \dots, 16$ when $n = 1$.

3.6.2 Constitutive theory for $\boldsymbol{q}$ of order n

In this derivation we consider the argument tensors of $\boldsymbol{q}$ in (3.93). We express $\boldsymbol{q}$ as a linear combination of the combined generators of the argument tensors of $\boldsymbol{q}$. As in the case of ${}_d\boldsymbol{\sigma}^{[0]}$, here also the coefficients in the linear combination are functions of the combined invariants of the argument tensors of $\boldsymbol{q}$. The material coefficients in this case are also derived using Taylor series expansion of each coefficient in the linear combination about a known configuration $\underline{\Omega}$. $\boldsymbol{q}$ is a tensor of rank one, but $\boldsymbol{\varepsilon}_{[0]}, \boldsymbol{\varepsilon}_{[i]};\ i = 1, 2, \dots, n$ are symmetric tensors of rank two and $\boldsymbol{g}$, θ are tensors of rank one and zero. Let ${}^{q}\underset{\sim}{\boldsymbol{G}}^i;\ i = 1, 2, \dots, \widetilde{N}$ be the combined generators of the argument tensors of $\boldsymbol{q}$ that are tensors of rank one and let ${}^{q\sigma}\underset{\sim}{I}^j;\ j = 1, 2, \dots, M$ be the combined invariants of the same argument tensors of $\boldsymbol{q}$, which are the same as those for ${}_d\boldsymbol{\sigma}^{[0]}$. Then, we can express $\boldsymbol{q}$ in the current configuration as a linear combination of ${}^q\underset{\sim}{\boldsymbol{G}}^i;\ i = 1, 2, \dots, \widetilde{N}$:

$$\boldsymbol{q} = -\sum_{i=1}^{\widetilde{N}} {}^q\underset{\sim}{\alpha}^i\, {}^q\underset{\sim}{\boldsymbol{G}}^i \tag{3.100}$$

The reasons for the absence of a constant term in (3.100) and the negative sign are well known [1–3, 58, 59].

$${}^q\underset{\sim}{\alpha}^i = {}^q\underset{\sim}{\alpha}^i\big({}^{q\sigma}\underset{\sim}{I}^j;\ j = 1, 2, \dots, M, \theta\big) \tag{3.101}$$

To determine the material coefficients from ${}^q\underset{\sim}{\alpha}^i;\ i = 1, 2, \dots, \widetilde{N}$, we consider Taylor series expansion of each ${}^q\underset{\sim}{\alpha}^i;\ i = 1, 2, \dots, \widetilde{N}$ about a known configuration $\underline{\Omega}$ in invariants ${}^{q\sigma}\underset{\sim}{I}^j;\ j = 1, 2, \dots, M$ and temperature θ and retain only up to linear terms in the invariants and the temperature θ.

$${}^q\underset{\sim}{\alpha}^i = {}^q\underset{\sim}{\alpha}^i\big|_{\underline{\Omega}} + \sum_{j=1}^{M} \left.\frac{\partial\, {}^q\underset{\sim}{\alpha}^i}{\partial\, {}^{q\sigma}\underset{\sim}{I}^j}\right|_{\underline{\Omega}} \big({}^{q\sigma}\underset{\sim}{I}^j - ({}^{q\sigma}\underset{\sim}{I}^j)_{\underline{\Omega}}\big) + \left.\frac{\partial\, {}^q\underset{\sim}{\alpha}^i}{\partial \theta}\right|_{\underline{\Omega}} \big(\theta - \theta_{\underline{\Omega}}\big);\ i = 1, 2, \dots, \widetilde{N} \tag{3.102}$$

When we substitute (3.102) into (3.100), we obtain the most general constitutive theory of order n for $\boldsymbol{q}$. Details of the derivation and the material coefficients are given in the following.

Substituting from (3.102) into (3.100)

$$\boldsymbol{q} = -\sum_{i=1}^{\widetilde{N}} \left({}^q\underset{\sim}{\alpha}^i\big|_{\underline{\Omega}} + \sum_{j=1}^{M} \left.\frac{\partial\, {}^q\underset{\sim}{\alpha}^i}{\partial\, {}^{q\sigma}\underset{\sim}{I}^j}\right|_{\underline{\Omega}} \big({}^{q\sigma}\underset{\sim}{I}^j - ({}^{q\sigma}\underset{\sim}{I}^j)_{\underline{\Omega}}\big) + \left.\frac{\partial\, {}^q\underset{\sim}{\alpha}^i}{\partial \theta}\right|_{\underline{\Omega}} \big(\theta - \theta_{\underline{\Omega}}\big) \right) {}^q\underset{\sim}{\boldsymbol{G}}^i \tag{3.103}$$

Collecting coefficients (only those defined in the known configuration $\underline{\Omega}$) of ${}^q\underset{\sim}{\boldsymbol{G}}^i$, ${}^{q\sigma}\underset{\sim}{I}^j\, {}^q\underset{\sim}{\boldsymbol{G}}^i$, and $(\theta - \theta_{\underline{\Omega}})^{q}\underset{\sim}{\boldsymbol{G}}^{i}$ in (3.103) and defining the material coefficients

$$
\begin{aligned}
{}^{q}\underline{b}_{i} &= {}^{q}\underset{\sim}{\alpha}^{i}\big|_{\underline{\Omega}} + \sum_{j=1}^{M} \left.\frac{\partial\, {}^{q}\underset{\sim}{\alpha}^{i}}{\partial\, {}^{q\sigma}\underset{\sim}{I}^{j}}\right|_{\underline{\Omega}} \; ; \; i = 1, 2, \dots, \widetilde{N} \\
{}^{q}\underline{c}_{ij} &= \left.\frac{\partial\, {}^{q}\underset{\sim}{\alpha}^{i}}{\partial\, {}^{q\sigma}\underset{\sim}{I}^{j}}\right|_{\underline{\Omega}} \; ; \; i = 1, 2, \dots, \widetilde{N}; \; j = 1, 2, \dots, M \\
{}^{q}\underline{d}_{i} &= \left.\frac{\partial\, {}^{q}\underset{\sim}{\alpha}^{i}}{\partial \theta}\right|_{\underline{\Omega}} \; ; \; i = 1, 2, \dots, \widetilde{N}
\end{aligned}
\tag{3.104}
$$

Equation (3.103) can be written as

$$
\begin{aligned}
\boldsymbol{q} = -\sum_{i=1}^{\widetilde{N}} {}^{q}\underline{b}_{i}\, {}^{q}\underset{\sim}{\boldsymbol{G}}^{i} - \sum_{i=1}^{\widetilde{N}} \sum_{j=1}^{M} {}^{q}\underline{c}_{ij}\, {}^{q\sigma}\underset{\sim}{I}^{j}\, {}^{q}\underset{\sim}{\boldsymbol{G}}^{i} \\
- \sum_{i=1}^{\widetilde{N}} {}^{q}\underline{d}_{i} (\theta - \theta_{\underline{\Omega}})\, {}^{q}\underset{\sim}{\boldsymbol{G}}^{i}
\end{aligned}
\tag{3.105}
$$

This constitutive theory for $\boldsymbol{q}$ requires $(\widetilde{N} + \widetilde{N}M + \widetilde{N})$ material coefficients defined in (3.104). The material coefficients defined in (3.104) are functions of invariants ${}^{q\sigma}\underset{\sim}{I}^{j}|_{\underline{\Omega}}$; $j = 1, 2, \dots, M$ and temperature $\theta|_{\underline{\Omega}}$ in the known configuration $\underline{\Omega}$. This constitutive theory is also based on integrity, hence is complete.

Remarks. (1) Using the general derivations presented for rate theories of order n for ${}_{d}\boldsymbol{\sigma}^{[0]}$ and $\boldsymbol{q}$, constitutive theories of any desired order can be obtained.

(2) Constitutive theory for $\boldsymbol{q}$ is consistent with the constitutive theory for ${}_{d}\boldsymbol{\sigma}^{[0]}$ as it uses the same argument tensors as in the case of ${}_{d}\boldsymbol{\sigma}^{[0]}$. At this stage, there is no rationale to alter the argument tensors of $\boldsymbol{q}$. The constitutive theory for $\boldsymbol{q}$ demonstrates the influence of $\boldsymbol{\varepsilon}$ and $\boldsymbol{\varepsilon}_{[i]}$; $i = 1, 2, \dots, n$ as well as their interaction with $\boldsymbol{g}$ on heat conduction. In a later section we consider a simpler constitutive theory for $\boldsymbol{q}$ to demonstrate this point more clearly. Table 3 gives combined generators ${}^{q}\underset{\sim}{\boldsymbol{G}}^{i}$; $i = 1, 2, \dots, 7$ when $n = 1$. The combined invariants ${}^{q\sigma}\underset{\sim}{I}^{j}|_{\underline{\Omega}}$; $j = 1, 2, \dots, 16$ for this case are the same as those in table 2.

3.6.3 Simplified rate constitutive theories of up to order n for ${}_d\boldsymbol{\sigma}^{[0]}$

Consider the rate constitutive theory of order n derived in section 5.1. We consider simplified rate theories of order n based on the following assumptions:

(1) The constitutive theories are linear in the components of each of the argument tensors of ${}_d\boldsymbol{\sigma}^{[0]}$.

(2) We neglect all terms containing the products of the components of the argument tensors, *i.e.*, the terms containing the products of the generators and the invariants as well as the terms containing products of $(\theta - \theta_{\underline{\Omega}})$ with the generators and the invariants.

Based on these two assumptions, the constitutive theories for ${}_d\boldsymbol{\sigma}^{[0]}$ will only contain tensors $\boldsymbol{\varepsilon}_{[i]}$; $i = 0, 1, \dots, n$ as generators and their traces as invariants, and we can write the following for ${}_d\boldsymbol{\sigma}^{[0]}$:

$$
\begin{aligned}
{}_d\boldsymbol{\sigma}^{[0]} = \underline{\sigma}^0|_{\underline{\Omega}}\boldsymbol{I} + a_1\boldsymbol{\varepsilon}_{[0]} + a_2\big(\operatorname{tr}\boldsymbol{\varepsilon}_{[0]}\big)\boldsymbol{I} + \sum_{i=1}^{n} b_i^1\boldsymbol{\varepsilon}_{[i]} \\
+ \sum_{i=1}^{n} b_i^2\big(\operatorname{tr}\boldsymbol{\varepsilon}_{[i]}\big)\boldsymbol{I} - \underline{\alpha}_{tm}|_{\underline{\Omega}}(\theta - \theta_{\underline{\Omega}})\boldsymbol{I}
\end{aligned}
\tag{3.106}
$$

The material coefficients a_1, a_2, b_i^1 and b_i^2 are functions of the invariants and temperature θ in configuration $\underline{\Omega}$. We can write (3.106) in the following form using Voigt's notation (in the absence of the first and last terms in (3.106)).

$$
\{{}_d\sigma^{[0]}\} = [\underline{a}]\{(\varepsilon_{[0]})\} + \sum_{i=1}^{n}[\underline{b}_i]\{(\varepsilon_{[i]})\} \tag{3.107}
$$

in which

$$
\{{}_d\sigma^{[0]}\}^T = [\,{}_d\sigma^{[0]}_{x_1x_1},\ {}_d\sigma^{[0]}_{x_2x_2},\ {}_d\sigma^{[0]}_{x_3x_3},\ {}_d\sigma^{[0]}_{x_2x_3},\ {}_d\sigma^{[0]}_{x_3x_1},\ {}_d\sigma^{[0]}_{x_1x_2}] \tag{3.108}
$$

and

$$[\underset{\sim}{a}] = \begin{bmatrix} a_1 + a_2 & a_2 & a_2 & 0 & 0 & 0 \\ a_2 & a_1 + a_2 & a_2 & 0 & 0 & 0 \\ a_2 & a_2 & a_1 + a_2 & 0 & 0 & 0 \\ 0 & 0 & 0 & a_1 & 0 & 0 \\ 0 & 0 & 0 & 0 & a_1 & 0 \\ 0 & 0 & 0 & 0 & 0 & a_1 \end{bmatrix} \tag{3.109}$$

The coefficients of $[\underset{\sim}{b}_i]$ can be obtained by using (3.109) and replacing a_1 and a_2 with b_i^1 and b_i^2. In (3.107), ${}_d\sigma^{[0]}$ is a linear function of the tensors $\varepsilon_{[i]};\ i = 0, 1, \dots, n$, hence the rate theories defined by (3.107) are simplified rate theories. In (3.107), $n = 0, 1, \dots, n$ yield simplified linear rate theories of orders $0, 1, \dots, n$ for ${}_d\sigma^{[0]}$.

3.6.4 Simplified rate theories of up to orders n for heat vector $\boldsymbol{q}$

In this section, we consider a simplified form of the rate constitutive theories of orders up to n for heat vector $\boldsymbol{q}$ in which $\boldsymbol{q}$ is a linear function of the components of tensors $\varepsilon_{[i]};\ i = 0, 1, \dots, n$ but the terms containing the products of the components of those tensors are neglected. All nonlinear terms in $\boldsymbol{g}$ are retained. Thus, in these theories, we only consider the following generators:

$$\boldsymbol{g},\ \varepsilon_{[i]}\boldsymbol{g};\ i = 0, 1, \dots, n \tag{3.110}$$

and the following invariants:

$$\boldsymbol{g} \cdot \boldsymbol{g},\ \operatorname{tr} \varepsilon_{[i]};\ i = 0, 1, \dots, n;\ \boldsymbol{g} \cdot (\varepsilon_{[i]}\boldsymbol{g});\ i = 0, 1, \dots, n \tag{3.111}$$

in the current configuration after Taylor series expansion of the coefficients ${}^q\alpha^i$ about a known configuration $\underline{\Omega}$, keeping in mind that these theories are linear in the argument tensors of $\boldsymbol{q}$ except $\boldsymbol{g}$. We can write the following for $\boldsymbol{q}$:

$$
\begin{aligned}
\boldsymbol{q} = & -\widetilde{a}\boldsymbol{g} - \sum_{i=0}^{n} c_i \boldsymbol{\varepsilon}_{[i]}\boldsymbol{g} - \underset{\sim}{a}\big(\boldsymbol{g}\cdot\boldsymbol{g}\big)\boldsymbol{g} \\
& - \sum_{i=0}^{n} \underset{\sim}{c}_i\big(\operatorname{tr}\boldsymbol{\varepsilon}_{[i]}\big)\boldsymbol{g} - \sum_{i=0}^{n} e_i\big(\boldsymbol{g}\cdot(\boldsymbol{\varepsilon}_{[i]}\boldsymbol{g})\big)\boldsymbol{g} - \underset{\sim}{f}(\theta-\theta_{\underline{\Omega}})\boldsymbol{g}
\end{aligned}
\tag{3.112}
$$

The coefficients $\widetilde{a}$, c_i, $\underset{\sim}{a}$, $\underset{\sim}{c}_i$, e_i, and $\underset{\sim}{f}$ are functions of the invariants and temperature θ in the known configuration $\underline{\Omega}$. In (3.112), $n = 0,1,\dots,n$ yield simplified linear rate theories of orders $0,1,\dots,n$ for heat vector $\boldsymbol{q}$.

3.6.5 Constitutive theory of order one ($n = 1$) for ${}_d\boldsymbol{\sigma}^{[0]}$

In this section we consider details of the constitutive theory of order one ($n = 1$) for deviatoric second Piola-Kirchhoff stress tensor.

Consider

$$
{}_d\boldsymbol{\sigma}^{[0]} = {}_d\boldsymbol{\sigma}^{[0]}\big(\boldsymbol{\varepsilon}_{[0]}, \boldsymbol{\varepsilon}_{[1]}, \boldsymbol{g}, \theta\big)
$$

The combined generators ${}^{\sigma}\underset{\sim}{\boldsymbol{G}}^i;\ i = 1,2,\dots,12$ of the argument tensors $\boldsymbol{\varepsilon}_{[0]}$, $\boldsymbol{\varepsilon}_{[1]}$, and $\boldsymbol{g}$ that are symmetric tensors of rank two are listed in table 3.1 [31]. The combined invariants ${}^{q\sigma}\underset{\sim}{I}^j;\ j = 1,2,\dots,16$ of the same argument tensors are listed in table 3.2.

$$
{}_d\boldsymbol{\sigma}^{[0]} = {}^{\sigma}\underset{\sim}{\alpha}^0\boldsymbol{I} + \sum_{i=1}^{12} {}^{\sigma}\underset{\sim}{\alpha}^i\,{}^{\sigma}\underset{\sim}{\boldsymbol{G}}^i \tag{3.113}
$$

in which

$$
{}^{\sigma}\underset{\sim}{\alpha}^i = {}^{\sigma}\underset{\sim}{\alpha}^i\big({}^{q\sigma}\underset{\sim}{I}^j;\ j = 1,2,\dots,16,\theta\big) \tag{3.114}
$$

To determine material coefficients from ${}^{\sigma}\underset{\sim}{\alpha}^i;\ i = 0,1,\dots,12$, we consider Taylor series expansion of each ${}^{\sigma}\underset{\sim}{\alpha}^i;\ i = 0,1,\dots,12$ in ${}^{q\sigma}\underset{\sim}{I}^j;\ j = 1,2,\dots,16$ and θ about a known

Table 3.1: Combined generators of $\boldsymbol{\varepsilon}_{[0]}$, $\boldsymbol{\varepsilon}_{[1]}$, and $\boldsymbol{g}$ that are symmetric tensors of rank two

i	${}^{\sigma}\underset{\sim}{\boldsymbol{G}}{}^{i}$
1	$\boldsymbol{\varepsilon}_{[0]}$
2	$(\boldsymbol{\varepsilon}_{[0]})^2$
3	$\boldsymbol{\varepsilon}_{[1]}$
4	$(\boldsymbol{\varepsilon}_{[1]})^2$
5	$\boldsymbol{\varepsilon}_{[0]}\boldsymbol{\varepsilon}_{[1]} + \boldsymbol{\varepsilon}_{[1]}\boldsymbol{\varepsilon}_{[0]}$
6	$(\boldsymbol{\varepsilon}_{[0]})^2\boldsymbol{\varepsilon}_{[1]} + \boldsymbol{\varepsilon}_{[1]}(\boldsymbol{\varepsilon}_{[0]})^2$
7	$\boldsymbol{\varepsilon}_{[0]}(\boldsymbol{\varepsilon}_{[1]})^2 + (\boldsymbol{\varepsilon}_{[1]})^2\boldsymbol{\varepsilon}_{[0]}$
8	$\boldsymbol{g} \otimes \boldsymbol{g}$
9	$\boldsymbol{g} \otimes \boldsymbol{\varepsilon}_{[0]}\boldsymbol{g} + \boldsymbol{\varepsilon}_{[0]}\boldsymbol{g} \otimes \boldsymbol{g}$
10	$\boldsymbol{g} \otimes (\boldsymbol{\varepsilon}_{[0]})^2\boldsymbol{g} + (\boldsymbol{\varepsilon}_{[0]})^2\boldsymbol{g} \otimes \boldsymbol{g}$
11	$\boldsymbol{g} \otimes \boldsymbol{\varepsilon}_{[1]}\boldsymbol{g} + \boldsymbol{\varepsilon}_{[1]}\boldsymbol{g} \otimes \boldsymbol{g}$
12	$\boldsymbol{g} \otimes (\boldsymbol{\varepsilon}_{[1]})^2\boldsymbol{g} + (\boldsymbol{\varepsilon}_{[1]})^2\boldsymbol{g} \otimes \boldsymbol{g}$

Table 3.2: Combined scalar invariants of $\boldsymbol{\varepsilon}_{[0]}$, $\boldsymbol{\varepsilon}_{[1]}$, and $\boldsymbol{g}$

j	${}^{q\sigma}\underset{\sim}{I}{}^{j}$
1	$\mathrm{tr}\,\boldsymbol{\varepsilon}_{[0]}$
2	$\mathrm{tr}(\boldsymbol{\varepsilon}_{[0]})^2$
3	$\mathrm{tr}\,\boldsymbol{\varepsilon}_{[0]}{}^3$
4	$\mathrm{tr}\,\boldsymbol{\varepsilon}_{[1]}$
5	$\mathrm{tr}(\boldsymbol{\varepsilon}_{[1]})^2$
6	$\mathrm{tr}(\boldsymbol{\varepsilon}_{[1]})^3$
7	$\mathrm{tr}\,\boldsymbol{\varepsilon}_{[0]}\boldsymbol{\varepsilon}_{[1]}$
8	$\mathrm{tr}(\boldsymbol{\varepsilon}_{[0]})^2\boldsymbol{\varepsilon}_{[1]}$
9	$\mathrm{tr}\,\boldsymbol{\varepsilon}_{[0]}(\boldsymbol{\varepsilon}_{[1]})^2$
10	$\mathrm{tr}(\boldsymbol{\varepsilon}_{[0]})^2(\boldsymbol{\varepsilon}_{[1]})^2$
11	$\boldsymbol{g} \cdot \boldsymbol{\varepsilon}_{[0]}\boldsymbol{g}$
12	$\boldsymbol{g} \cdot (\boldsymbol{\varepsilon}_{[0]})^2\boldsymbol{g}$
13	$\boldsymbol{g} \cdot \boldsymbol{\varepsilon}_{[1]}\boldsymbol{g}$
14	$\boldsymbol{g} \cdot (\boldsymbol{\varepsilon}_{[1]})^2\boldsymbol{g}$
15	$\boldsymbol{g} \cdot \boldsymbol{\varepsilon}_{[0]}\boldsymbol{\varepsilon}_{[1]}\boldsymbol{g}$
16	$\boldsymbol{g} \cdot \boldsymbol{g}$

configuration $\underline{\Omega}$ and retain only up to linear terms in ${}^{q\sigma}\underset{\sim}{\underline{I}}^j$; $j = 1, 2, \ldots, 16$ and θ.

$$
{}^{\sigma}\underset{\sim}{\underline{\alpha}}^i = {}^{\sigma}\underset{\sim}{\underline{\alpha}}^i|_{\underline{\Omega}} + \sum_{j=1}^{16} \left. \frac{\partial\, {}^{\sigma}\underset{\sim}{\underline{\alpha}}^i}{\partial\, {}^{q\sigma}\underset{\sim}{\underline{I}}^j} \right|_{\underline{\Omega}} \left({}^{q\sigma}\underset{\sim}{\underline{I}}^j - ({}^{q\sigma}\underset{\sim}{\underline{I}}^j)_{\underline{\Omega}} \right) + \left. \frac{\partial\, {}^{\sigma}\underset{\sim}{\underline{\alpha}}^i}{\partial \theta} \right|_{\underline{\Omega}} (\theta - \theta_{\underline{\Omega}});\; i = 0, 1, \ldots, 12 \tag{3.115}
$$

We substitute (3.115) into (3.113) and collect coefficients (those in configuration $\underline{\Omega}$) of the same terms as shown in section 5.1 and follow the same procedure to obtain the final form of the constitutive theory that is exactly the same as (3.99) with material coefficients defined in (3.98), except that in this case, $N = 12$ and $M = 16$. This constitutive theory, though of first order, still requires 233 material coefficients.

3.6.5.1 Simplifications

Consider a constitutive theory for ${}_d\boldsymbol{\sigma}^{[0]}$ that is linear in the components of $\boldsymbol{\varepsilon}_{[0]}$, $\boldsymbol{\varepsilon}_{[1]}$, and $\boldsymbol{g}$ and neglects all product terms between tensors $\boldsymbol{\varepsilon}_{[0]}$, $\boldsymbol{\varepsilon}_{[1]}$, $\boldsymbol{g}$, and θ in the current configuration.

(1) In this theory, all generators except $\boldsymbol{\varepsilon}_{[0]}$ and $\boldsymbol{\varepsilon}_{[1]}$ are neglected.

(2) All invariants in the current configuration except $\mathrm{tr}(\boldsymbol{\varepsilon}_{[0]})$ and $\mathrm{tr}(\boldsymbol{\varepsilon}_{[1]})$ are neglected as well.

(3) The material coefficients may still have dependence on the desired invariants (in the known configuration $\underline{\Omega}$) even though some of these may have been neglected in the current configuration in the construction of the constitutive theory.

With these assumptions, the rate constitutive theory of order one ($n = 1$) reduces to

$$
\begin{aligned}
{}_d\boldsymbol{\sigma}^{[0]} &= \underline{\sigma}^0|_{\underline{\Omega}}\boldsymbol{I} + a_1\boldsymbol{\varepsilon}_{[0]} + a_2\,\mathrm{tr}(\boldsymbol{\varepsilon}_{[0]})\boldsymbol{I} \\
&\quad + b_1\boldsymbol{\varepsilon}_{[1]} + b_2\,\mathrm{tr}(\boldsymbol{\varepsilon}_{[1]})\boldsymbol{I} - (\underline{\alpha}_{tm})_{\underline{\Omega}}(\theta - \theta_{\underline{\Omega}})\boldsymbol{I}
\end{aligned} \tag{3.116}
$$

The material coefficients a_1, a_2, b_1, b_2, and $(\underline{\alpha}_{tm})_{\underline{\Omega}}$ can be functions of the desired invariants of the tensors $\boldsymbol{\varepsilon}_{[0]}$ and $\boldsymbol{\varepsilon}_{[1]}$ (and others if so desired) and temperature θ in the configuration $\underline{\Omega}$. In this constitutive theory, deviatoric second Piola-Kirchhoff stress is a linear function of Green's strain $\boldsymbol{\varepsilon}_{[0]}$ and the rate of Green's strain. In (3.116), the terms with coefficients a_1 and a_2 are responsible for deviatoric Cauchy stress tensor due to the strain field, whereas the terms with coefficients b_1 and b_2 are related to the strain rate dependent deviatoric stress field and are responsible for the dissipation mechanism. This is shown more clearly for the 1-d case and a comparison with the Kelvin-Voigt model. For simplicity, but without loss of generality, we consider (3.116) in the absence of the first and last terms on the right side.

$$
{}_{d}\boldsymbol{\sigma}^{[0]} = a_1 \boldsymbol{\varepsilon}_{[0]} + a_2 \,\mathrm{tr}(\boldsymbol{\varepsilon}_{[0]}) \boldsymbol{I} + b_1 \boldsymbol{\varepsilon}_{[1]} + b_2 \,\mathrm{tr}(\boldsymbol{\varepsilon}_{[1]}) \boldsymbol{I} \tag{3.117}
$$

We can also write (3.117) using Voigt's notation:

$$
\{ {}_{d}\sigma^{[0]} \} = [\underset{\sim}{a}] \{ (\varepsilon_{[0]}) \} + [\underset{\sim}{b}] \{ (\varepsilon_{[1]}) \} \tag{3.118}
$$

in which

$$
\{ {}_{d}\sigma^{[0]} \}^T = [\, {}_{d}\sigma^{[0]}_{x_1 x_1}, \, {}_{d}\sigma^{[0]}_{x_2 x_2}, \, {}_{d}\sigma^{[0]}_{x_3 x_3}, \, {}_{d}\sigma^{[0]}_{x_2 x_3}, \, {}_{d}\sigma^{[0]}_{x_3 x_1}, \, {}_{d}\sigma^{[0]}_{x_1 x_2}] \tag{3.119}
$$

The components of $\{ (\varepsilon_{[0]}) \}$ and $\{ (\varepsilon_{[1]}) \}$ in (3.107) are arranged in the same fashion as those of $\{ {}_{d}\sigma^{[0]} \}$ in (3.108), and

$$
[\underset{\sim}{a}] = \begin{bmatrix}
a_1 + a_2 & a_2 & a_2 & 0 & 0 & 0 \\
a_2 & a_1 + a_2 & a_2 & 0 & 0 & 0 \\
a_2 & a_2 & a_1 + a_2 & 0 & 0 & 0 \\
0 & 0 & 0 & a_1 & 0 & 0 \\
0 & 0 & 0 & 0 & a_1 & 0 \\
0 & 0 & 0 & 0 & 0 & a_1
\end{bmatrix} \tag{3.120}
$$

The components of $\underset{\sim}{\boldsymbol{b}}$ can be obtained by replacing a with b in (3.120).

Remarks. (1) The coefficients of matrix $[\underset{\sim}{a}]$ are completely defined using material coefficients a_1 and a_2. For linear elastic solids, $\underset{\sim}{a}_{ij}$ of $[\underset{\sim}{a}]$ are functions of modulus of elasticity E and Poisson's ratio ν or Lamé's constants.

(2) For elastic materials, the strain rate in one direction must be accompanied by the strain rate in the two mutually orthogonal directions as well, due to Poisson's effect. This is evident in the composition of the $[\underset{\sim}{b}]$ matrix in which the (3×3) portion of the $[\underset{\sim}{b}]$ matrix associated with normal strain rates is fully populated, (*i.e.*, not diagonal).

(3) Thus, we see that the mathematical model associated with strain rate dependent dissipation requires two material coefficients (just like strain dependent elasticity), b_1 and b_2 in this case.

(4) When we consider the 1-d case for incompressible viscoelastic solid matter and compare with the Kelvin-Voigt model, we shall observe that dissipation is due to the second term on the right side of (3.107).

(5) From (3.107), we note that if we ignore the first term on the right side of (3.107), then the reduced (3.107) is similar to Newton's law of viscosity, but in Lagrangian description, which confirms that the mechanism of dissipation in the theories presented here is the same as viscous fluids.

3.6.5.2 *Further simplifications: 1-d case for incompressible viscoelastic solids and comparison with Kelvin-Voigt model*

If we consider the 1-d case of (3.107) (using x_1 coordinate axis), then we can write

$$
{}_{d}\sigma^{[0]}_{x_1 x_1} = (a_1 + a_2)(\varepsilon_{[0]})_{x_1 x_1} + (b_1 + b_2)(\varepsilon_{[1]})_{x_1 x_1} \tag{3.121}
$$

In order to compare this constitutive model with the Kelvin-Voigt model, we consider infinitesimal deformation for which case Green's strain tensor reduces to the symmetric part of the displacement gradient tensor (${}^{l}\boldsymbol{\varepsilon}_{[0]}$), and its rate becomes ${}^{l}\boldsymbol{\varepsilon}_{[1]}$. Furthermore, the distinction between different measures of stress disappears and we can simply denote the deviatoric part of the stress tensor by ${}_{d}\boldsymbol{\sigma}$. Thus, (3.121) reduces to (using $\underset{\sim}{a}_{11} = a_1 + a_2$ and $\underset{\sim}{b}_{11} = b_1 + b_2$)

$$ {}_{d}\sigma_{x_1x_1} = \underset{\sim}{a}_{11}({}^{l}\varepsilon_{[0]})_{x_1x_1} + \underset{\sim}{b}_{11}({}^{l}\varepsilon_{[1]})_{x_1x_1} \tag{3.122} $$

in which $({}^{l}\varepsilon_{[0]})_{x_1x_1}$ is defined by

$$ ({}^{l}\varepsilon_{[0]})_{x_1x_1} = \frac{\partial u_{x_1}}{\partial x_1} \tag{3.123} $$

and

$$ ({}^{l}\varepsilon_{[1]})_{x_1x_1} = \frac{\partial}{\partial t}\left(\frac{\partial u_{x_1}}{\partial x_1}\right) \tag{3.124} $$

where u_{x_1} is the displacement in the x_1 direction.

Substituting from (3.123) and (3.124) into (3.121), we finally have

$$ {}_{d}\sigma_{x_1x_1} = \underset{\sim}{a}_{11}\frac{\partial u_{x_1}}{\partial x_1} + \underset{\sim}{b}_{11}\frac{\partial}{\partial t}\left(\frac{\partial u_{x_1}}{\partial x_1}\right) \tag{3.125} $$

The classical Kelvin-Voigt constitutive model originates with the simple assumption that the mechanisms of elasticity and dissipation in incompressible viscoelastic solids without memory for the simple one-dimensional case can be viewed as being due to a spring and a dashpot arranged in parallel. The stress due to the spring is proportional to the strain, while the stress due to the dashpot is considered to be proportional to the strain rate. Then, for the one-dimensional case (infinitesimal deformation), we can write

$$ \sigma_{x_1x_1} = c_1({}^{l}\varepsilon_{[0]})_{x_1x_1} + c_2({}^{l}\varepsilon_{[1]})_{x_1x_1} \tag{3.126} $$

in which $\sigma_{x_1x_1}$ is a measure of total stress and $({}^{l}\varepsilon_{[0]})_{x_1x_1}$ and $({}^{l}\varepsilon_{[1]})_{x_1x_1}$ are measures of strain and strain rates associated with infinitesimal deformation. Based on published work [46, 50], (3.126) holds for incompressible viscoelastic solid matter as a special 1-d case.

Substituting from (3.123) and (3.124) into (3.126)

$$\sigma_{x_1x_1} = c_1 \frac{\partial u_{x_1}}{\partial x_1} + c_2 \frac{\partial}{\partial t}\left(\frac{\partial u_{x_1}}{\partial x_1}\right). \tag{3.127}$$

If we set $\underset{\sim}{a}_{11}$ to c_1 and $\underset{\sim}{b}_{11}$ to c_2 in (3.125), then the right side of (3.125) is exactly the same as the right side of (3.127) from the 1-d Kelvin-Voigt model. The major difference between (3.125) and (3.127) is that (3.125) is a constitutive theory for deviatoric stress ${}_d\sigma_{x_1x_1}$, whereas (3.127) is for the total stress $\sigma_{x_1x_1}$.

It is instructive to consider the momentum equation in Lagrangian description for the 1-d case under consideration. In the absence of body forces, we have

$$\rho_0 \frac{\partial^2 u_{x_1}}{\partial t^2} - \frac{\partial p}{\partial x_1} + \frac{\partial({}_d\sigma_{x_1x_1})}{\partial x_1} = 0 \tag{3.128}$$

For incompressible thermoviscoelastic solids, the mechanical pressure is in fact the mean normal stress, which in the 1-d case reduces to [1]

$$p = -\frac{1}{2}\,{}_d\sigma_{x_1x_1} \tag{3.129}$$

Substituting from (3.129) into (3.128)

$$\rho_0 \frac{\partial^2 u_{x_1}}{\partial t^2} + \frac{3}{2}\frac{\partial(\,{}_d\sigma_{x_1x_1})}{\partial x_1} = 0 \tag{3.130}$$

Substituting for ${}_d\sigma_{x_1x_1}$ from (3.125) into (3.130)

$$\rho_0 \frac{\partial^2 u_{x_1}}{\partial t^2} + \left(\frac{3}{2}\underset{\sim}{a}_{11}\right)\frac{\partial^2 u_{x_1}}{\partial x_1^2} + \left(\frac{3}{2}\underset{\sim}{b}_{11}\right)\frac{\partial}{\partial x_1}\left(\frac{\partial}{\partial t}\left(\frac{\partial u_{x_1}}{\partial x_1}\right)\right) = 0 \tag{3.131}$$

or

$$\rho_0 \frac{\partial^2 u_{x_1}}{\partial t^2} + \left(\frac{3}{2}\underset{\sim}{a}_{11}\right)\frac{\partial^2 u_{x_1}}{\partial x_1^2} + \left(\frac{3}{2}\underset{\sim}{b}_{11}\right)\frac{\partial}{\partial t}\left(\frac{\partial^2 u_{x_1}}{\partial x_1^2}\right) = 0 \tag{3.132}$$

Equation (3.132) is the momentum or equilibrium equation for the special 1-d case resulting from the use of the constitutive theory presented in this chapter. In the absence of the third term in (3.132), we have the equilibrium equation for the 1-d linear elastic case with infinitesimal deformation.

On the other hand, if we do not use the stress decomposition, then the momentum equation (3.128) can be written as

$$\rho_0 \frac{\partial^2 u_{x_1}}{\partial t^2} + \frac{\partial \sigma_{x_1 x_1}}{\partial x_1} = 0 \tag{3.133}$$

Substituting from (3.127) into (3.133)

$$\rho_0 \frac{\partial^2 u_{x_1}}{\partial t^2} + c_1 \frac{\partial^2 u_{x_1}}{\partial x_1^2} + c_2 \frac{\partial}{\partial t}\left(\frac{\partial^2 u_{x_1}}{\partial x_1^2}\right) = 0 \tag{3.134}$$

If we define $\frac{3}{2}\underset{\sim}{a}_{11} = c_1$ and $\frac{3}{2}\underset{\sim}{b}_{11} = c_2$, then (3.133) and (3.134) are identical for this special 1-d case of incompressible viscoelastic medium with infinitesimal strain tensor.

Remarks. (1) The simple first order rate theory when further simplified and applied to 1-d incompressible viscoelastic solid with infinitesimal deformation yields a constitutive theory for deviatoric axial stress that is similar in appearance to the Kelvin-Voigt model but is quite different due to the fact that in the Kelvin-Voigt model, the constitutive theory is for total axial stress.

(2) The derivation of the n^{th} order rate theory shows that when strain and strain rate tensors are arguments of the stress tensor, decomposition of the stress tensor into equilibrium and deviatoric tensors is essential. The constitutive theories for equilibrium stress tensor for compressible as well as incompressible cases are derived using

the conditions resulting from the entropy inequality, while the constitutive theory for deviatoric stress tensor is derived using theory of generators and invariants. These derivations are consistent with the axioms and principles of continuum mechanics.

(3) Based on (2) and the derivation presented for the simplified 1-d case, it is straightforward to conclude that the 1-d Kelvin-Voigt model for viscoelastic solids is not supported by the rate theories presented in this chapter.

(4) For the 1-d linear elastic incompressible case with infinitesimal deformation, the momentum equation or equilibrium equation resulting from the theories presented here and the Kelvin-Voigt model are the same, provided the material coefficients are assumed to be identical, which is not the case.

(5) In the 1-d Kelvin-Voigt model, a single material coefficient describes the mechanism of dissipation (damping coefficient for the dashpot). From the derivation presented here and the physics, it is quite clear that the mechanism of dissipation in viscoelastic solids requires two material coefficients.

(6) The 1-d spring and dashpot in parallel is a phenomenological model that has no mechanism for its extension: (i) to 2-d or 3-d or continuous media in general, (ii) for compressible matter, (iii) for finite deformation and finite strain, whereas the derivations presented in this work are for finite deformation, valid in $\mathbb{R}$, $\mathbb{R}^2$, and $\mathbb{R}^3$, and are consistent with the axioms of constitutive theory in continuum mechanics.

3.6.6 Constitutive theory of order one ($n = 1$) for $\boldsymbol{q}$

For rate constitutive theory of order one for the heat vector $\boldsymbol{q}$, we have

$$\boldsymbol{q} = \boldsymbol{q}(\boldsymbol{\varepsilon}_{[0]}, \boldsymbol{\varepsilon}_{[1]}, \boldsymbol{g}, \theta) \tag{3.135}$$

The combined generators ${}^q\underset{\sim}{\boldsymbol{G}}^i;\ i = 1, 2, \ldots, 7$ of the argument tensors $\boldsymbol{\varepsilon}_{[0]}$, $\boldsymbol{\varepsilon}_{[1]}$, and $\boldsymbol{g}$

Table 3.3: Combined generators of $\boldsymbol{\varepsilon}_{[0]}$, $\boldsymbol{\varepsilon}_{[1]}$, and $\boldsymbol{g}$ for tensors of rank one

i	${}^{q}\underset{\sim}{\boldsymbol{G}}^{i}$
1	$\boldsymbol{g}$
2	$\boldsymbol{\varepsilon}_{[0]}\boldsymbol{g}$
3	$(\boldsymbol{\varepsilon}_{[0]})^2\boldsymbol{g}$
4	$\boldsymbol{\varepsilon}_{[1]}\boldsymbol{g}$
5	$(\boldsymbol{\varepsilon}_{[1]})^2\boldsymbol{g}$
6	$\boldsymbol{\varepsilon}_{[0]}\boldsymbol{\varepsilon}_{[1]}\boldsymbol{g}$
7	$\boldsymbol{\varepsilon}_{[1]}\boldsymbol{\varepsilon}_{[0]}\boldsymbol{g}$

that are tensors of rank one are given in table 3.3. The combined invariants ${}^{q\sigma}\underset{\sim}{I}^{j}$; $j = 1,2,\ldots,16$ of the argument tensors $\boldsymbol{\varepsilon}_{[0]}$, $\boldsymbol{\varepsilon}_{[1]}$, and $\boldsymbol{g}$ remain the same as those listed in table 3.2. Using the combined generators ${}^{q}\underset{\sim}{\boldsymbol{G}}^{i}$; $i = 1,2,\ldots,7$, we can write

$$\boldsymbol{q} = -\sum_{i=1}^{7} {}^{q}\alpha^{i}\,{}^{q}\underset{\sim}{\boldsymbol{G}}^{i} \tag{3.136}$$

The coefficients of ${}^{q}\alpha^{i}$ in (3.136) are functions of the invariants ${}^{q\sigma}\underset{\sim}{I}^{j}$; $j = 1,2,\ldots,16$ and temperature θ in the current configuration. To determine the material coefficients using ${}^{q}\alpha^{i}$; $i = 1,2,\ldots,7$, we consider Taylor series expansion of the coefficients ${}^{q}\alpha^{i}$; $i = 1,2,\ldots,7$ in ${}^{q\sigma}\underset{\sim}{I}^{j}$; $j = 1,2,\ldots,16$ and θ about a known configuration $\underline{\Omega}$ and retain only up to linear terms in the invariants ${}^{q\sigma}\underset{\sim}{I}^{j}$ and the temperature θ.

$${}^{q}\alpha^{i} = {}^{q}\alpha^{i}\big|_{\underline{\Omega}} + \sum_{j=1}^{16} \left.\frac{\partial\,{}^{q}\alpha^{i}}{\partial\,{}^{q\sigma}\underset{\sim}{I}^{j}}\right|_{\underline{\Omega}} \left({}^{q\sigma}\underset{\sim}{I}^{j} - ({}^{q\sigma}\underset{\sim}{I}^{j})_{\underline{\Omega}}\right) + \left.\frac{\partial\,{}^{q}\alpha^{i}}{\partial\theta}\right|_{\underline{\Omega}} (\theta - \theta_{\underline{\Omega}});\ i = 1,2,\ldots,7 \tag{3.137}$$

When ${}^{q}\alpha^{i}$; $i = 1,2,\ldots,7$ are substituted in (3.136), we obtain the constitutive theory of order one ($n = 1$) for heat vector $\boldsymbol{q}$. Details of the derivation follow the same steps as in section 3.6.2 (rate theory of order n) and we obtain the final expression for $\boldsymbol{q}$ that is exactly the same as equation (3.105) except that in this case $\widetilde{N} = 7$ and $M = 16$. This constitutive theory requires 126 material coefficients.

3.6.6.1 Simplified first order theories for $\boldsymbol{q}$

Choice of selected generators and invariants

The constitutive theory of order one for $\boldsymbol{q}$ contains too many material constants. In this section, we consider a simplified constitutive theory for $\boldsymbol{q}$ in which $\boldsymbol{q}$ is a linear function of the components of $\boldsymbol{\varepsilon}_{[0]}$ and $\boldsymbol{\varepsilon}_{[1]}$, and the terms containing products of $\boldsymbol{\varepsilon}_{[0]}$ and $\boldsymbol{\varepsilon}_{[1]}$ are neglected. In additions, we neglect products of $(\theta - \theta_{\underline{\Omega}})$ with all other terms except $\boldsymbol{g}$. Thus, in this theory we only retain generators

$$\boldsymbol{g}, \quad \boldsymbol{\varepsilon}_{[0]}\boldsymbol{g}, \quad \boldsymbol{\varepsilon}_{[1]}\boldsymbol{g} \tag{3.138}$$

and invariants

$$\boldsymbol{g}\cdot\boldsymbol{g}, \quad \operatorname{tr}\boldsymbol{\varepsilon}_{[0]}, \quad \operatorname{tr}\boldsymbol{\varepsilon}_{[1]}, \quad \boldsymbol{g}\cdot\left(\boldsymbol{\varepsilon}_{[0]}\boldsymbol{g}\right), \quad \boldsymbol{g}\cdot\left(\boldsymbol{\varepsilon}_{[1]}\boldsymbol{g}\right) \tag{3.139}$$

in the current configuration after Taylor series expansion of the coefficients ${}^{q}\alpha^{i}$ about a known configuration $\underline{\Omega}$, keeping in mind that this theory is linear in components of $\boldsymbol{\varepsilon}_{[0]}$ and $\boldsymbol{\varepsilon}_{[1]}$

$$\begin{aligned} \boldsymbol{q} = &-\widetilde{a}_1\boldsymbol{g} - \widetilde{a}_5\boldsymbol{\varepsilon}_{[0]}\boldsymbol{g} - \widetilde{a}_6\boldsymbol{\varepsilon}_{[1]}\boldsymbol{g} \\ &- \widetilde{a}_2(\boldsymbol{g}\cdot\boldsymbol{g})\boldsymbol{g} - \widetilde{a}_3(\operatorname{tr}\boldsymbol{\varepsilon}_{[0]})\boldsymbol{g} - \widetilde{a}_4(\operatorname{tr}\boldsymbol{\varepsilon}_{[1]})\boldsymbol{g} \\ &- \widetilde{a}_7(\boldsymbol{g}\cdot(\boldsymbol{\varepsilon}_{[0]}\boldsymbol{g}))\boldsymbol{g} - \widetilde{a}_8(\boldsymbol{g}\cdot(\boldsymbol{\varepsilon}_{[1]}\boldsymbol{g}))\boldsymbol{g} \\ &- \widetilde{a}_9(\theta - \theta_{\underline{\Omega}})\boldsymbol{g} \end{aligned} \tag{3.140}$$

By regrouping terms

$$\begin{aligned} \boldsymbol{q} = -\Bigg(&\Big(\widetilde{a}_1 + \widetilde{a}_2\boldsymbol{g}\cdot\boldsymbol{g} + \widetilde{a}_3\operatorname{tr}\boldsymbol{\varepsilon}_{[0]} + \widetilde{a}_4\operatorname{tr}\boldsymbol{\varepsilon}_{[1]} + \boldsymbol{g}\cdot\left(\widetilde{a}_7\boldsymbol{\varepsilon}_{[0]} + \widetilde{a}_8\boldsymbol{\varepsilon}_{[1]}\right)\boldsymbol{g}\Big)\boldsymbol{I} \\ &+ \widetilde{a}_5\boldsymbol{\varepsilon}_{[0]} + \widetilde{a}_6\boldsymbol{\varepsilon}_{[1]} + \Bigg)\boldsymbol{g} - \widetilde{a}_9(\theta - \theta_{\underline{\Omega}})\boldsymbol{g} \end{aligned} \tag{3.141}$$

The coefficients $\widetilde{a}_i$; $i = 1, 2, \ldots, 9$ are functions of the invariants ${}^{q\sigma}\underset{\sim}{I}^{j}$; $j = 1, 2, \ldots, 16$ and temperature θ. This constitutive theory illustrates more explicitly the influence of $\boldsymbol{\varepsilon}_{[0]}$

and $\boldsymbol{\varepsilon}_{[1]}$ on the heat vector $\boldsymbol{q}$.

Remarks. (1) If we consider a constitutive theory in which we assume $\boldsymbol{\varepsilon}_{[0]}$ and $\boldsymbol{\varepsilon}_{[1]}$ not to influence $\boldsymbol{q}$, then the terms containing $\boldsymbol{\varepsilon}_{[0]}$ and $\boldsymbol{\varepsilon}_{[1]}$ in (3.141) can be deleted and we have

$$\boldsymbol{q} = -(\widetilde{a}_1 + \widetilde{a}_2 \boldsymbol{g}\cdot\boldsymbol{g})\boldsymbol{g} - \widetilde{a}_9(\theta - \theta_{\underline{\Omega}})\boldsymbol{g} \tag{3.142}$$

Material coefficients $\widetilde{a}_1$, $\widetilde{a}_2$ and $\widetilde{a}_9$ are functions of $(\boldsymbol{g}\cdot\boldsymbol{g})|_{\underline{\Omega}}$ and $\theta|_{\underline{\Omega}}$.

(2) If we consider a constitutive theory that is linear in $\boldsymbol{g}$, then (3.141) reduces to

$$\begin{aligned}\boldsymbol{q} = -\Big(\widetilde{a}_1 + \Big(\widetilde{a}_3\big(\operatorname{tr}\boldsymbol{\varepsilon}_{[0]}\big)\boldsymbol{I} + \widetilde{a}_5\boldsymbol{\varepsilon}_{[0]}\Big) + \Big(\widetilde{a}_4\big(\operatorname{tr}\boldsymbol{\varepsilon}_{[1]}\big)\boldsymbol{I}\\ + \widetilde{a}_6\boldsymbol{\varepsilon}_{[1]}\Big)\Big)\boldsymbol{g} - \widetilde{a}_9(\theta - \theta_{\underline{\Omega}})\boldsymbol{g}\end{aligned} \tag{3.143}$$

This constitutive theory illustrates the influence of strain and strain rate on $\boldsymbol{q}$.

Using reduced argument tensors for q

If we delete $\boldsymbol{\varepsilon}_{[j]};\ j = 0, 1, \ldots, n$ from the arguments of $\boldsymbol{q}$, then we have

$$\boldsymbol{q} = \boldsymbol{q}(\boldsymbol{g}, \theta) \tag{3.144}$$

Derivation of the constitutive theory based on (3.144) using the theory of generators and invariants has been presented in Chapter 2. The final form of the constitutive theory resulting from this approach is exactly the same as (3.142).

Constitutive theory for q using entropy inequality

The derivation of the constitutive theory using this approach results in the well-known

Fourier heat conduction law

$$q_j(\boldsymbol{g}) = -k_{ij}(\theta) g_i \tag{3.145}$$

This derivation assumes that $\boldsymbol{q}$ is proportional to $-\boldsymbol{g}$.

3.6.6.2 Remarks on theories for $\boldsymbol{q}$ in section 3.6.6.1

(1) The simplified first order constitutive theory for $\boldsymbol{q}$ in 3.6.6.1 based on a limited number of generators and invariants is by far the most complete theory out of the theories in (3.141) – (3.143), *i.e.*, (3.142) and (3.143) are obviously subsets of (3.141). The constitutive theory (3.141), though simplified, clearly demonstrates the influence of strain rates and interactions of strain rates and $\boldsymbol{g}$ on the heat vector $\boldsymbol{q}$.

(2) If we neglect the influence of strain rates on $\boldsymbol{q}$, then (3.142) (same as theory in section 3.6.6.1) is the most complete constitutive theory for $\boldsymbol{q}$. This theory accounts for the fact that $\boldsymbol{q}$ is an inherently nonlinear function of $\boldsymbol{g}$ and that the material coefficients can be functions of $(\boldsymbol{g} \cdot \boldsymbol{g})_{\underline{\Omega}}$ and $\theta|_{\underline{\Omega}}$.

(3) The Fourier heat conduction law derived directly from the entropy inequality (section 3.6.6.1) is perhaps the simplest but also the most approximate constitutive theory for $\boldsymbol{q}$. It only permits the material coefficient to be a function of $\theta|_{\underline{\Omega}}$. $\boldsymbol{q}$ as a linear function of $\boldsymbol{g}$ in this theory is by inherent assumption.

3.7 Numerical studies

In this section, we present some numerical studies using the constitutive theory presented here and comparisons with the currently and commonly model for dissipation in solids in which dissipation is assumed to be proportional to velocity. We consider the 1-d case with infinitesimal deformation. As shown earlier for this particular 1-d case, the momentum equation resulting from the constitutive theory presented here and using the Kelvin-Voigt

1-d model is the same (equation (3.132) and (3.134))

$$\rho_0 \frac{\partial^2 u_{x_1}}{\partial t^2} + c_1 \frac{\partial^2 u_{x_1}}{\partial x_1^2} + c_2 \frac{\partial}{\partial t}\left(\frac{\partial^2 u_{x_1}}{\partial x_1^2}\right) = 0 \quad \forall (x_1, t) \in \begin{array}{l} \Omega_{x_1,t} = \\ \Omega_{x_1} \times \Omega_t = \\ \Omega_{x_1} \times [0, \tau] \end{array} \tag{3.146}$$

If we assume dissipation is proportional to velocity (a commonly used mechanism for structural damping in solids [60]), then the momentum equation becomes

$$\rho_0 \frac{\partial^2 u_{x_1}}{\partial t^2} + c_1 \frac{\partial^2 u_{x_1}}{\partial x_1^2} + \underset{\sim}{c}_2 \frac{\partial u_{x_1}}{\partial t} = 0 \quad \forall (x_1, t) \in \Omega_{x_1,t} \tag{3.147}$$

We remark that in the derivation of (3.147) we only have constitutive theory for elastic behavior; the damping or dissipation is directly through the momentum equation (rate of change of momentum must be equal to the sum of the forces acting on the volume of matter) in terms of a dissipative force (the third term in (3.147)).

Using (3.146) and (3.147) as mathematical models we present some numerical studies for 1-d wave propagation using space-time least squares finite element processes based on a space-time strip $\Omega_{x_1} \times \Delta t$ for an increment of time with time marching. The local approximations for a space-time element are considered in higher order space-time scalar product spaces that permit higher-order global differentiability approximations in space and time.

First, we derive the dimensionless form of the PDEs (3.146) and (3.147). Consider PDE (3.146). We rewrite (3.146) by introducing a hat ($\widehat{\ }$) on all quantities signifying that all quantities have their usual dimensions

$$\widehat{\rho}_0 \frac{\partial^2 \widehat{u}_{x_1}}{\partial \widehat{t}^2} + \widehat{c}_1 \frac{\partial^2 \widehat{u}_{x_1}}{\partial \widehat{x}_1^2} + \widehat{c}_2 \frac{\partial}{\partial \widehat{t}}\left(\frac{\partial^2 \widehat{u}_{x_1}}{\partial \widehat{x}_1^2}\right) = 0 \quad \forall (\widehat{x}_1, \widehat{t}) \in \begin{array}{l} \Omega_{\widehat{x}_1,\widehat{t}} = \\ \Omega_{\widehat{x}_1} \times \Omega_{\widehat{t}} = \\ (0, \widehat{L}) \times (0, \widehat{\tau}) \end{array} \tag{3.148}$$

We choose reference quantities with subscript $_0$ or $_{ref}$ and define dimensionless variables using these.

Let

$$\rho_0 = \frac{\widehat{\rho}_0}{(\rho_0)_{ref}},\ u_{x_1} = \frac{\widehat{u}_{x_1}}{u_0},\ v_{x_1} = \frac{\widehat{v}_{x_1}}{v_0},\ x_1 = \frac{\widehat{x}_1}{L_0},\ t = \frac{\widehat{t}}{t_0} \tag{3.149}$$

L_0 is a reference length, hence $u_0 = L_0$, and if we choose v_0 as a reference velocity (generally the speed of sound using reference quantities), then $t_0 = \frac{u_0}{v_0} = \frac{L_0}{v_0}$

$$\rho_0 \frac{\partial^2 u_{x_1}}{\partial t^2} + \left(\frac{\widehat{c}_1}{(\rho_0)_{ref} v_0^2} \right) \frac{\partial^2 u_{x_1}}{\partial x_1^2} + \left(\frac{\widehat{c}_2}{L_0 (\rho_0)_{ref} v_0} \right) \frac{\partial}{\partial t} \left(\frac{\partial^2 u_{x_1}}{\partial x_1^2} \right) = 0 \quad \forall (x_1, t) \in \Omega_{x_1,t} \tag{3.150}$$

$(\rho_0)_{ref} v_0^2$ has dimensions of stress; we generally define $\tau_0 = (\rho_0)_{ref} v_0^2$ as the reference stress based on characteristic kinetic energy. Likewise, $L_0 (\rho_0)_{ref} v_0$ has units of viscosity and is essentially the Reynolds number with a unit reference viscosity.

We define

$$c_1^d = \frac{\widehat{c}_1}{(\rho_0)_{ref} v_0^2} \text{ and } c_2^d = \frac{\widehat{c}_2}{L_0 (\rho_0)_{ref} v_0} \tag{3.151}$$

Then the dimensionless form of (3.146) using (3.150) becomes

$$\rho_0 \frac{\partial^2 u_{x_1}}{\partial t^2} + c_1^d \frac{\partial^2 u_{x_1}}{\partial x_1^2} + c_2^d \frac{\partial}{\partial t} \left(\frac{\partial^2 u_{x_1}}{\partial x_1^2} \right) = 0 \quad \forall (x_1, t) \in \Omega_{x_1,t} : \qquad \text{Model A} \tag{3.152}$$

Following the same procedure, we can also derive the dimensionless form of (3.147):

$$\rho_0 \frac{\partial^2 u_{x_1}}{\partial t^2} + c_1^d \frac{\partial^2 u_{x_1}}{\partial x_1^2} + \underset{\sim}{c}_2^d \frac{\partial u_{x_1}}{\partial t} = 0 \quad \forall (x_1, t) \in \Omega_{x_1,t} : \qquad \text{Model B} \tag{3.153}$$

in which

$$\underset{\sim}{c}_2^d = \frac{\underset{\sim}{c}_2 L_0}{(\rho_0)_{ref} v_0} \tag{3.154}$$

In the numerical studies, we consider an axial rod of dimensionless length one unit and choose $(\rho_0)_{ref} = \widehat{\rho}_0$ so that ρ_0 in (3.152) and (3.153) becomes unity. The spatial do-

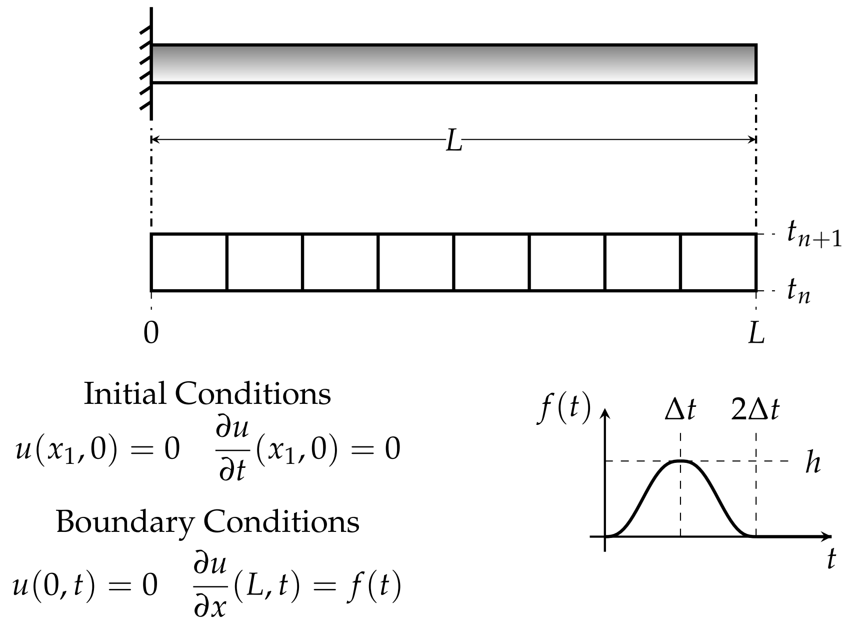

Figure 3.1: Schematic diagram for the studied problem

main $[0, 1]$ for an increment of time $\Delta t = 0.1$, *i.e.*, the space-time strip $[0, L] \times [0, \Delta t]$, is discretized using a uniform mesh of eight nine-node, p-version, higher order global differentiability, space-time finite elements [52–57]. Figure 3.1 shows details of the space-time domain for a time increment $\Delta t = t_{n+1} - t_n$ and boundary conditions as well as the initial conditions. The left end of the rod is clamped (impermeable boundary) and the right end is subjected to a continuous and differentiable strain distribution, such that the strain is twice continuously differentiable with respect to time, over a time period of $2\Delta t$. For $t \geq 2\Delta t$, the applied strain at the right end of the rod is zero. We choose a p-level of 9 in space and time and a local approximation of class $C^{11}(\bar{\Omega}^e_{x_1,t})$, *i.e.*, of class C^1 in space and time. The finite element formulations used for computing numerical solutions for both models (Models A and B) are based on space-time least squares process constructed using residual functionals [53–57]. The resulting computational process is unconditionally stable. Evolutions are computed using a space-time strip with time marching [53–57]. For the choice of local approximation ($C^{11}(\bar{\Omega}^e_{x_1,t})$), the integrals in the finite element processes are Lebesgue, but due to the smoothness of the evolution for the 8-element discretization with a p-level of 9, the residual functionals are of the order of $O(10^{-6})$ or lower for all space-time strips for both model problems, confirming good accuracy of evolution. For both model problems, the evolutions are completed for $0 \leq t \leq 2.0$. In both model problems, we choose $c_1^d = 1.0$. Since c_2^d and $\underset{\sim}{c}_2^d$ in model problems A and B do not have the same meaning (*i.e.*, physics), a direct comparison of the evolutions for the same values of c_2^d and $\underset{\sim}{c}_2^d$ is not meaningful. For this reason, we choose a range of values for c_2^d and $\underset{\sim}{c}_2^d$ to show behaviors of dissipation in models A and B.

Figures 3.2 and 3.3 show plots of strain $\varepsilon_{x_1x_1}$ versus x_1 for different values of time for $c_2^d = 0$, 0.1, and 0.5 for model A. Figures 3.4 and 3.5 show plots of strain $\varepsilon_{x_1x_1}$ versus x_1 for model B for different values of time using $\underset{\sim}{c}_2^d = 0$, 1.0, and 2.0.

From figures 3.2 and 3.3 for model A, we note that when $c_2^d = 0$, we have pure elastic strain wave propagation. Reflections of the strain wave at the fixed and free boundaries are simulated perfectly. Since in this case there is no dissipation, the wave shape is preserved during propagation, *i.e.*, no amplitude decay or base elongation is observed. At the fixed boundary, the amplitude of the strain wave doubles as expected, and the reflected wave at the free boundary returns as a tensile wave. When c_2^d is nonzero, we observe amplitude decay and base elongation of the strain wave during the evolution. Progressively increasing values of c_2^d result in large amplitude decay and base elongation, due to increased dissipation. This mechanism of dissipation is exactly the same as viscous dissipation in fluids (as discussed earlier).

From figures 3.4 and 3.5, for model B, we note that for $\underset{\sim}{c}_2^d = 0$, we have exactly the same behavior as in figures 3.2 and 3.3 as in this case, the two models are identical. For nonzero values of $\underset{\sim}{c}_2^d$, the evolutions in figures 3.4 and 3.5 show amplitude decay of the strain wave, but the base of the wave is preserved during the evolution, regardless of the values of $\underset{\sim}{c}_2^d$. Thus, this mechanism of dissipation is quite different in model B from the case of model A. As expected, progressively increasing values of $\underset{\sim}{c}_2^d$ result in progressively decaying amplitudes of the strain wave, indicating progressively increased dissipation,

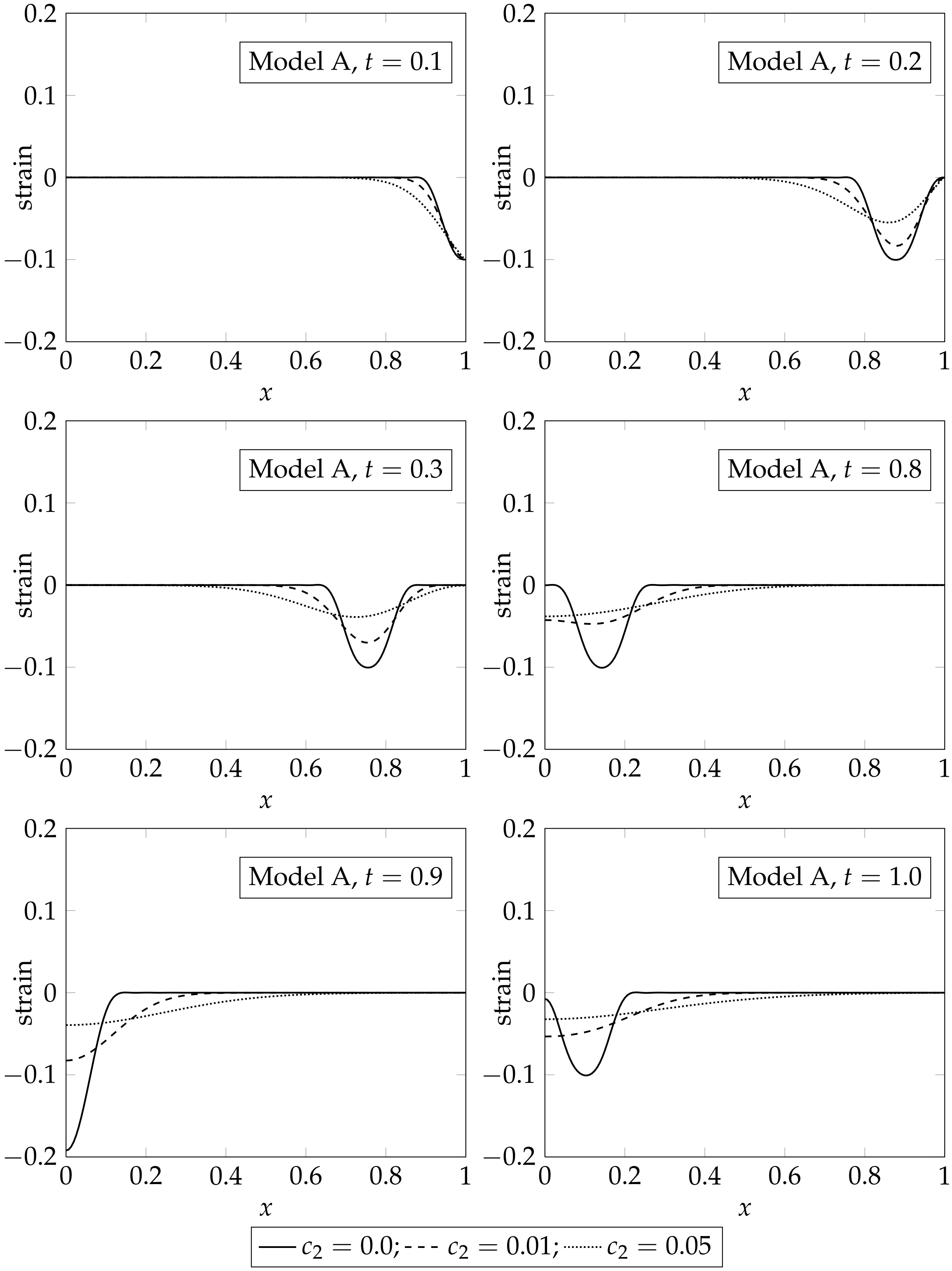

Figure 3.2: Time evolution of the propagation of an applied strain pulse in a one-dimensional axially-deforming rod with dissipation based on strain rate (model A), using three different values of the dissipation parameter c_2

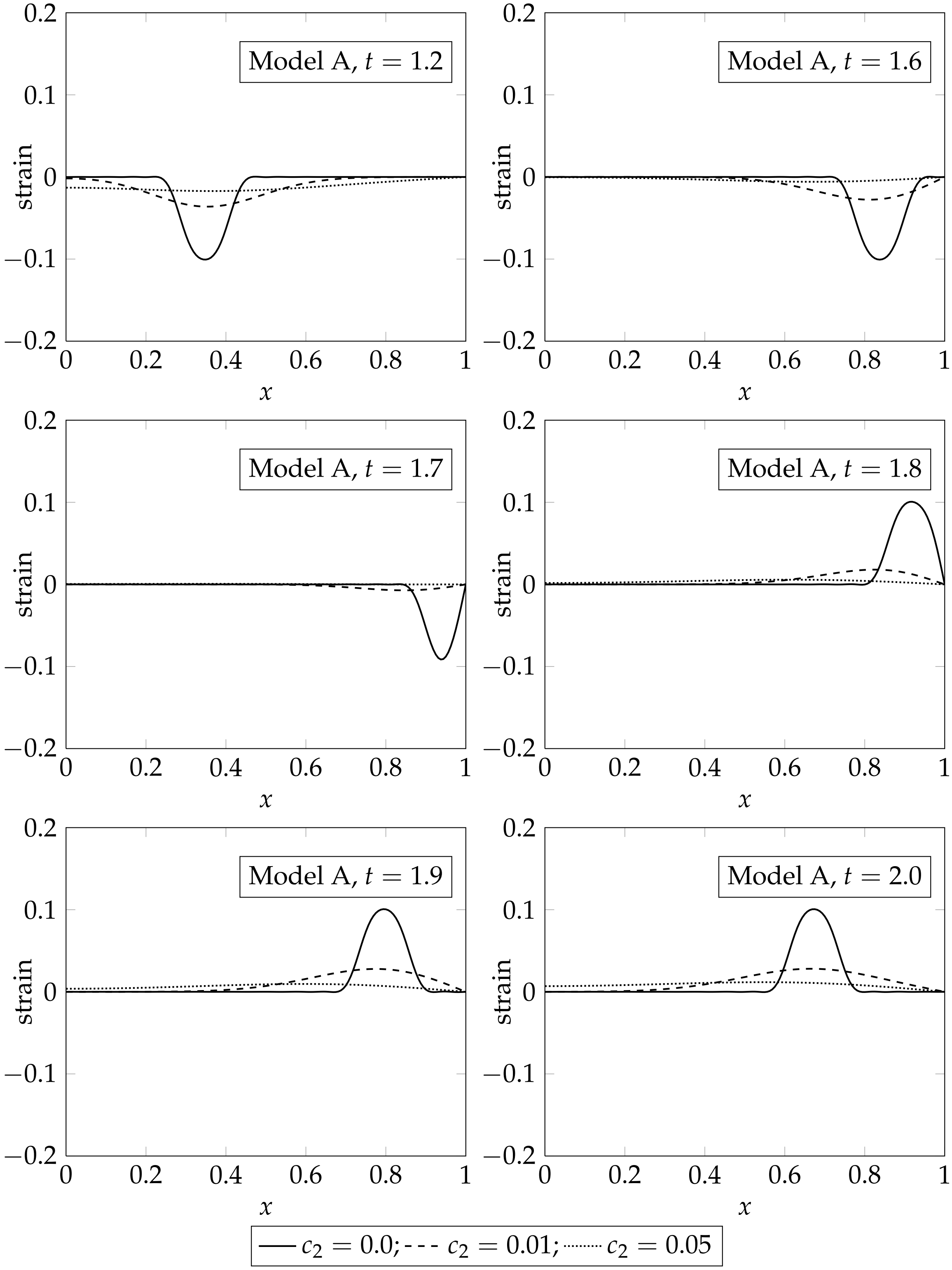

Figure 3.3: Continued time evolution of the propagation of an applied strain pulse in a one-dimensional axially-deforming rod with dissipation based on strain rate (model A), using three different values of the dissipation parameter c_2

but the base of the strain wave remains unaltered during the entire evolution.

3.8 Summary

In this work, rate constitutive theories of order n are derived using the theory of generators and invariants for stress tensor ${}_d\boldsymbol{\sigma}^{[0]}$ and heat vector $\boldsymbol{q}$ by using $\boldsymbol{\varepsilon}_{[i]};\ i = 0, 1, \ldots, n, \boldsymbol{g}$, and θ as argument tensors of ${}_d\boldsymbol{\sigma}^{[0]}$ and $\boldsymbol{q}$. It is shown that for thermoviscoelastic solids, the decomposition $\boldsymbol{\sigma}^{[0]} = {}_e\boldsymbol{\sigma}^{[0]} + {}_d\boldsymbol{\sigma}^{[0]}$ is essential and that the constitutive theory with dissipation mechanism due to each strain rate in $\boldsymbol{\varepsilon}_{[i]};\ i = 1, 2, \ldots, n$ is only due to ${}_d\boldsymbol{\sigma}^{[0]}$. The n^{th} order rate theories for ${}_d\boldsymbol{\sigma}^{[0]}$ and $\boldsymbol{q}$ are consistent in the choices of argument tensors.

Many simplified theories of the n^{th} order rate theories for ${}_d\boldsymbol{\sigma}^{[0]}$ and $\boldsymbol{q}$ are presented and compared with currently used constitutive theories such as the Kelvin-Voigt model and the Fourier heat conduction law. Details of the comparison (including numerical studies) and the conclusions drawn from these are given in various sections.

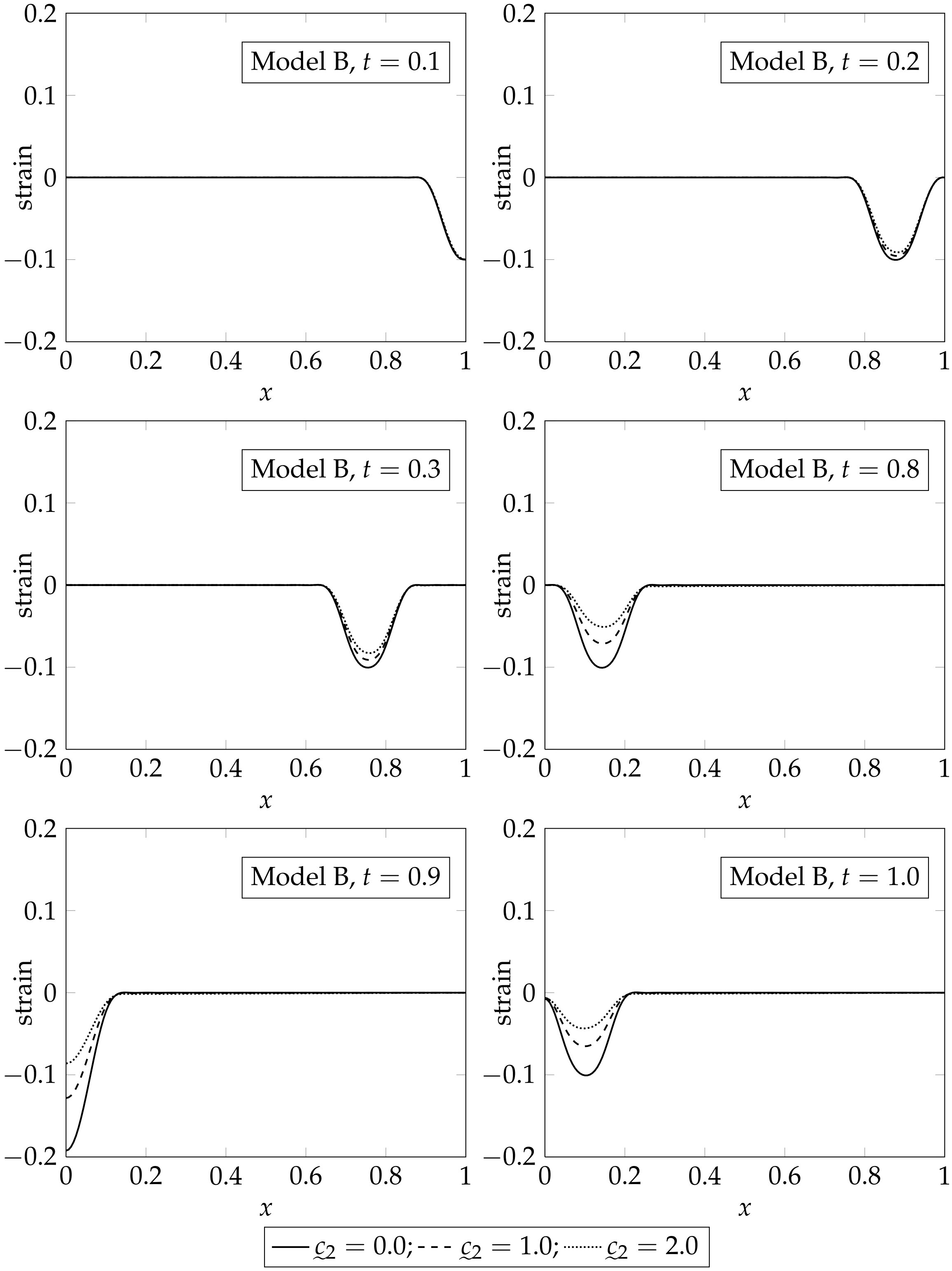

Figure 3.4: Continued time evolution of the propagation of an applied strain pulse in a one-dimensional axially-deforming rod with dissipation based on velocity (model B), using three different values of the dissipation parameter $\underset{\sim}{c}_2$

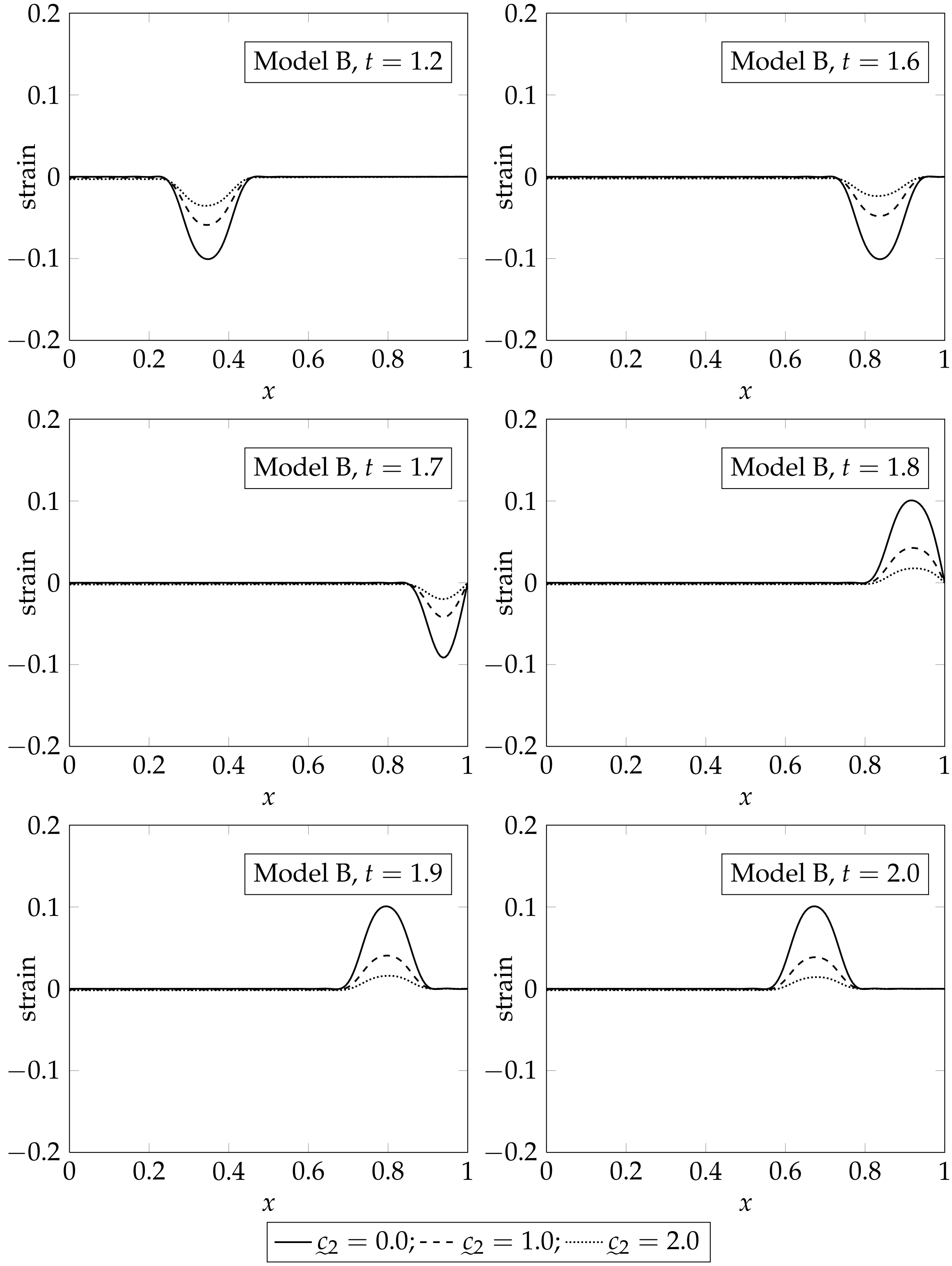

Figure 3.5: Continued time evolution of the propagation of an applied strain pulse in a one-dimensional axially-deforming rod with dissipation based on velocity (model B), using three different values of the dissipation parameter $\underset{\sim}{c}_2$

Chapter 4

Ordered rate constitutive theories in Lagrangian description for thermoviscoelastic solids with memory

4.1 Introduction

In the work presented here, we consider the derivation of constitutive theories for homogeneous, isotropic thermoviscoelastic solids with memory undergoing finite deformation during evolution. It is shown that the constitutive theories derived here can be theories of any desired orders. The constitutive theories are derived using the entropy inequality expressed in terms of the Helmholtz free energy density Φ through systematic development. From the entropy inequality in Φ, we establish Φ, entropy density η, second Piola-Kirchhoff stress $\boldsymbol{\sigma}^{[0]}$, and heat vector $\boldsymbol{q}$ as possible choices of dependent variables in the derivation of the constitutive theories at the onset of the development. The arguments of the dependent variables are established, beginning with $\boldsymbol{J}$, the fundamental measure of deformation, and also including $\boldsymbol{J}_{[i]};\ i = 1, 2, \ldots, n$ in addition to $\boldsymbol{J}$ as arguments of the dependent variables. At the very least, $\boldsymbol{J}_{[1]}$ is required as an argument to introduce a dissipation mechanism in the constitutive theories. The introduction of $\boldsymbol{J}_{[i]};\ i = 1, 2, \ldots, n$, compared to $\boldsymbol{J}_{[1]}$ only, is a generalization. It has been shown that to introduce the mechanism of memory in thermoviscoelastic fluids [59], at the very least $\bar{\boldsymbol{\sigma}}^{(0)}$ and $\bar{\boldsymbol{\sigma}}^{(1)}$ (contravariant convected time derivatives of orders zero and one of the contravariant Cauchy stress tensor) must be part of the constitutive theory for $\bar{\sigma}^{(0)}$. This concept was generalized [59] to include $\bar{\sigma}^{(m)}$ as a dependent variable in the constitutive theory, with $\bar{\sigma}^{(i)};\ i = 0, 1, \dots, m-1$ as its argument tensors as well as other dependent variables in the constitutive theories. In the present work, we use similar concepts but in Lagrangian description. Thus, now we have Φ, η, $\boldsymbol{\sigma}^{[m]}$, and $\boldsymbol{q}$ as dependent variables in the constitutive theories, with $\boldsymbol{\sigma}^{[i]};\ i = 0, 1, \dots, m-1$ and $\boldsymbol{J}_{[j]};\ j = 0, 1, \dots, n$ as their arguments. We also include $\boldsymbol{g}$ and θ in the arguments of the dependent variables. By following the axiom of frame invariance, a polar decomposition and dependence of $\boldsymbol{S}_r$ on $\boldsymbol{\varepsilon}_{[0]}$ and its generalization eventually leads to $\boldsymbol{\varepsilon}_{[0]}$ and $\boldsymbol{\varepsilon}_{[i]};\ i = 1, 2, \dots, n$ as argument tensors of the dependent variables instead of $\boldsymbol{J}$ and $\boldsymbol{J}_{[i]};\ i = 1, 2, \dots, n$. When $\dot{\Phi}$ (obtained using the chain rule of differentiation) is substituted into the entropy inequality, the choice of dependent variables in the constitutive theories reduces to Φ, $\boldsymbol{\sigma}^{[m]}$, and $\boldsymbol{q}$.

From the conditions resulting from the entropy inequality, there is no mechanism to derive the constitutive theories for $\boldsymbol{\sigma}^{[0]}$ and $\boldsymbol{q}$ using Φ, $\boldsymbol{\sigma}^{[m]}$, and $\boldsymbol{q}$ as dependent variables in the constitutive theories. Upon a decomposition of $\boldsymbol{\sigma}^{[0]}$ into equilibrium stress ${}_e\boldsymbol{\sigma}^{[0]}$ and deviatoric stress ${}_d\boldsymbol{\sigma}^{[0]}$, the conditions resulting from the entropy inequality are used to establish the constitutive theory for ${}_e\boldsymbol{\sigma}^{[0]}$ as thermodynamic pressure as a function of density and temperature for compressible matter and mechanical pressure as a function of temperature (after including the incompressibility constraint in the entropy inequality). After substituting the stress decomposition into the entropy inequality, the resulting conditions also require the work expended due to ${}_d\boldsymbol{\sigma}^{[0]}$ to be positive but provide no mechanism for deriving constitutive theories for ${}_d\boldsymbol{\sigma}^{[0]}$. We also have $q_i g_i \leq 0$ from the entropy inequality, which can be used to derive constitutive theory for $\boldsymbol{q}$.

In the work presented here, we will use the theory of generators and invariants [31] to derive constitutive theories for ${}_d\boldsymbol{\sigma}^{[0]}$ and $\boldsymbol{q}$. These constitutive theories contain material derivatives of $\boldsymbol{\varepsilon}$ and $\boldsymbol{\sigma}^{[0]}$ up to orders n and m, *i.e.*, rates of $\boldsymbol{\varepsilon}$ and $\boldsymbol{\sigma}^{[0]}$ up to orders n and m, hence they are referred to as ordered rate constitutive theories of orders (m, n). The constitutive theories for $\boldsymbol{q}$ are also derived using the conditions resulting from the entropy inequality ($q_i g_i \leq 0$) and using the theory of generators and invariants with reduced argument tensors. Many simplified forms of the ordered rate theories of orders (m, n) are considered and compared with currently used theories, such as the Zener model [61] to demonstrate consistency of the theories in the presented work and lack of continuum mechanics basis for constitutive models such as Zener. Numerical studies are also presented using space-time *hpk* finite element processes [52–57] for 1-d wave propagation. The work presented here shows that the mechanism of dissipation is due to $\boldsymbol{\varepsilon}_{[i]};\ i = 1, 2, \ldots, n$, thus at the very least, we must include $\boldsymbol{\varepsilon}_{[1]}$ (in addition to other argument tensors) to have a dissipation mechanism in the resulting constitutive theories. Likewise, the mechanism of memory requires that at the very least, $\boldsymbol{\sigma}^{[0]}$ and $\boldsymbol{\sigma}^{[1]}$ both must be included in the development of the constitutive theories for $\boldsymbol{\sigma}^{[0]}$. The derivation of the memory modulus is presented to show that such materials have fading memory.

4.2 Second law of thermodynamics, dependent variables in the rate constitutive theories, and their arguments

The second law of thermodynamics resulting in the entropy inequality must form the basis for deriving constitutive theories for all deforming matter to ensure thermodynamic equilibrium during the evolution of the deforming matter. We consider the entropy inequality in Lagrangian description derived in terms of Helmholtz free energy density Φ and the conjugate stress and strain pair $\boldsymbol{\sigma}^*$ and $\dot{\boldsymbol{J}}$. We can also use the entropy inequality in conjugate pair $\boldsymbol{\sigma}^{[0]}$ and $\dot{\varepsilon}$. Both forms of entropy inequality are precisely equivalent, as the conjugate pairs $\boldsymbol{\sigma}^*$,$\dot{\boldsymbol{J}}$ and $\boldsymbol{\sigma}^{[0]}$, $\dot{\varepsilon}$ are transformable from each other. Thus, when deriving constitutive theories, the choice of one form of entropy inequality over the other is immaterial. $\boldsymbol{\sigma}^*$ is the first Piola-Kirchhoff stress tensor, $\dot{\boldsymbol{J}}$ is the material derivative of the Jacobian of deformation $J_{ij} = \frac{\partial \bar{x}_i}{\partial x_j}$ where x_k and $\bar{x}_k$ are undeformed and deformed coordinates of a material point, $\boldsymbol{\sigma}^{[0]}$ is the second Piola-Kirchhoff stress tensor, and $\dot{\boldsymbol{\varepsilon}}$ is the material derivative of the Green's strain tensor. In this chapter, we consider entropy inequality in terms of Φ and $\boldsymbol{\sigma}^*$, $\dot{\boldsymbol{J}}$

$$\rho_0(\dot{\Phi} + \eta\dot{\theta}) + \frac{|J|q_i g_i}{\theta} - \sigma^*_{ik}\dot{J}_{ik} \leq 0 \tag{4.1}$$

where ρ_0 is the density in the undeformed configuration (also used as reference configuration). η is the entropy density, θ is the absolute temperature, $|J|$ is the determinant of the Jacobian of deformation, $\boldsymbol{q}$ is the heat vector, and $\boldsymbol{g}$ is the temperature gradient vector.

In the balance of momenta and the first law of thermodynamics, we assume the existence of a stress field and heat vector in the deforming matter during evolution, thus stress tensor and heat vector are naturally dependent variables related to the constitution of the matter; hence, they must be dependent variables in the constitutive theories. This is also supported by the entropy inequality (4.1). In addition, from (4.1), we note that Φ and η must also be considered as dependent variables in the constitutive theories, keeping in mind that some of these may be eliminated due to some other considerations if so warranted. θ, $\boldsymbol{J}$, $\dot{\boldsymbol{J}}$, and $\boldsymbol{g}$ cannot be dependent variables in the constitutive theories as either they are self observable or can be defined using self observable quantities such as temperature and material point displacements.

The Jacobian of deformation $\boldsymbol{J}$ is the fundamental measure of deformation, and hence, it must be considered as an argument of all dependent variables in the constitutive theories. In rate theories (as shown in reference [62]), all dependent variables in the constitutive theories must also exhibit dependence on the material derivatives of the Jacobian of deformation, *i.e.*, $\boldsymbol{J}_{[i]};\ i = 1, 2, \ldots, n$. These are linearly independent fundamental kinematic measures of rates of various orders up to n, *i.e.*, these are measures of rates up to order n. Hence, at the outset of the development of the rate theories, we also consider $\boldsymbol{J}_{[i]};\ i = 1, 2, \ldots, n$ as arguments of all dependent variables in the constitutive theories in addition to $\boldsymbol{J}$. From the development of the constitutive theories in for thermoviscoelastic fluids [59] and their simplifications resulting in Maxwell model, Oldroyd-B model, and Giesekus model [63], we know that for such fluids to have memory, at the very least, the constitutive theory must consider the first convected time derivative of the stress tensor as a dependent variable with stress and strain rate as its arguments (in addition to others). The generalization of this concept leads to the (m,n) ordered rate theories for thermoviscoelastic fluids [58,59]. In the case of solids, we utilize a similar concept but in Lagrangian description. This leads to the material derivative m of the conjugate stress tensor as a dependent variable in the constitutive theory as opposed to the stress tensor, and in addition to $\boldsymbol{J}$ and $\boldsymbol{J}_{[i]};\ i=1,2,\dots,n$ as arguments of all dependent variables, we now also consider stress tensor and its material derivatives up to order $m-1$ as arguments of the dependent variables in the constitutive theories. Additionally, we also consider θ and $\boldsymbol{g}$ as arguments of all dependent variables in the constitutive theories. Thus, in the rate constitutive theories considered here, the possible dependent variables at this stage are $\boldsymbol{\sigma}^{*[m]}$, $\boldsymbol{q}$, Φ, and η, and their argument tensors are $\boldsymbol{J}$, $\boldsymbol{J}_{[i]};\ i=1,2,\dots,n$, $\boldsymbol{\sigma}^{*[j]};\ j=0,1,\dots,m-1$, $\boldsymbol{g}$, and θ.

$$
\begin{aligned}
\Phi &= \Phi\big(\boldsymbol{\sigma}^{*[j]};\ j=0,1,\dots,m-1,\boldsymbol{J},\boldsymbol{J}_{[i]};\ i=1,2,\dots,n,\boldsymbol{g},\theta\big)\\
\boldsymbol{q} &= \boldsymbol{q}\big(\boldsymbol{\sigma}^{*[j]};\ j=0,1,\dots,m-1\boldsymbol{J},\boldsymbol{J}_{[i]};\ i=1,2,\dots,n,\boldsymbol{g},\theta\big)\\
\eta &= \eta\big(\boldsymbol{\sigma}^{*[j]};\ j=0,1,\dots,m-1,\boldsymbol{J},\boldsymbol{J}_{[i]};\ i=1,2,\dots,n,\boldsymbol{g},\theta\big)\\
\boldsymbol{\sigma}^{*[m]} &= \boldsymbol{\sigma}^{*[m]}\big(\boldsymbol{\sigma}^{*[j]};\ j=0,1,\dots,m-1,\boldsymbol{J},\boldsymbol{J}_{[i]};\ i=1,2,\dots,n,\boldsymbol{g},\theta\big)
\end{aligned}
\tag{4.2}
$$

We note that $\boldsymbol{J}_{[1]}$ in (4.2) and $\dot{\boldsymbol{J}}$ in (4.1) both represent the first material derivative of $\boldsymbol{J}$. Now, since the arguments of Φ are defined in (4.2), we can obtain a more explicit form of $\dot{\Phi}$ using Φ in (4.2).

$$
\dot{\Phi} = \frac{\partial\Phi}{J_{kl}}\dot{J}_{kl} + \sum_{i=1}^{n}\frac{\partial\Phi}{\partial(J_{[i]})_{kl}}(\dot{J}_{[i]})_{kl} + \sum_{j=0}^{m-1}\frac{\partial\Phi}{\partial(\sigma^{*[j]})_{kl}}(\dot{\sigma^{*}}^{[j]})_{kl} + \frac{\partial\Phi}{\partial\theta}\dot{\theta} + \frac{\partial\Phi}{\partial g_i}\dot{g}_i
\tag{4.3}
$$

Substituting from (4.3) into (4.1) and collecting coefficients

$$\left(\rho_0 \frac{\partial \Phi}{\partial J_{kl}} - \sigma^*_{lk}\right) \dot{J}_{kl} + \rho_0 \left(\frac{\partial \Phi}{\partial \theta} + \eta\right) \dot{\theta} + \sum_{i=1}^{n} \rho_0 \frac{\partial \Phi}{\partial (J_{[i]})_{kl}} (\dot{J}_{[i]})_{kl} + \sum_{j=0}^{m-1} \frac{\partial \Phi}{\partial (\sigma^{*[j]})_{kl}} (\dot{\sigma^*}^{[j]})_{kl} + \rho_0 \frac{\partial \Phi}{\partial g_i} \dot{g}_i + \frac{|J| q_i g_i}{\theta} \leq 0 \tag{4.4}$$

In order for (4.4) to hold for arbitrary but admissible $\dot{\boldsymbol{J}}_{[m]};\ m = 1, 2, \ldots, n,\ \boldsymbol{\sigma}^{*[j]};\ j = 0, 1, \ldots, m-1,\ \dot{\boldsymbol{g}}$, and $\dot{\theta}$, the following must hold:

$$\begin{aligned} \left(\rho_0 \frac{\partial \Phi}{\partial J_{kl}} - \sigma^*_{lk}\right) \dot{J}_{ik} + \frac{|J| q_i g_i}{\theta} &\leq 0 \\ \rho_0 \left(\frac{\partial \Phi}{\partial \theta} + \eta\right) &= 0 \\ \rho_0 \frac{\partial \Phi}{\partial (\dot{J}_{[i]})_{kl}} &= 0;\ i = 1, 2, \ldots, n \\ \rho_0 \frac{\partial \Phi}{\partial (\sigma^{*[j]})_{kl}} &= 0;\ j = 0, 1, \ldots, m-1 \\ \rho_0 \frac{\partial \Phi}{\partial g_i} &= 0 \end{aligned} \tag{4.5}$$

Since ρ_0 is constant, ρ_0 can be dropped from the last four equations in (4.5), and we have:

$$\left(\rho_0 \frac{\partial \Phi}{\partial J_{kl}} - \sigma^*_{lk}\right) \dot{J}_{ik} + \frac{|J| q_i g_i}{\theta} \leq 0 \tag{4.6}$$

$$\left(\frac{\partial \Phi}{\partial \theta} + \eta\right) = 0 \tag{4.7}$$

$$\frac{\partial \Phi}{\partial (\dot{J}_{[i]})_{kl}} = 0;\ i = 1, 2, \ldots, n \tag{4.8}$$

$$\frac{\partial \Phi}{\partial (\sigma^{*[j]})_{kl}} = 0;\ j = 0, 1, \ldots, m-1 \tag{4.9}$$

$$\frac{\partial \Phi}{\partial g_i} = 0 \tag{4.10}$$

Remarks. (1) From equation (4.7), we have $\eta = -\frac{\partial \Phi}{\partial \theta}$, hence η is deterministic from Φ, therefore η is not a dependent variable in the constitutive theories.

(2) Equation (4.8) implies that Φ is not a function of $\boldsymbol{J}_{[i]};\ i = 1, 2, \dots, n$.

(3) Equation (4.9) implies that Φ is not a function of $\boldsymbol{\sigma}^{*[i]};\ i = 0, 1, \dots, m-1$ either

(4) Equation (4.10) implies that Φ is not a function of $\boldsymbol{g}$ also.

(5) The inequality in (4.6) is essential in the form it is stated. For example,

$$\rho_0 \frac{\partial \Phi}{\partial J_{ik}} - \sigma^*_{ki} = 0 \text{ and } \frac{|J| q_i g_i}{\theta} \leq 0 \tag{4.11}$$

is inappropriate due to the fact that these imply that $\boldsymbol{\sigma}^*$ is not a function of $\boldsymbol{J}_{[i]};\ i = 1, 2, \dots, n$, as Φ is not a function of $\boldsymbol{J}_{[i]};\ i = 1, 2, \dots, n$, which is contrary to (4.2). We note that (4.6) in the stated form is unable to provide further details or mechanism for deriving constitutive theories for $\boldsymbol{\sigma}^*$ and $\boldsymbol{q}$.

(6) Based on remarks 1-5 (4.2) reduces to

$$\begin{aligned} \Phi &= \Phi\big(\boldsymbol{J}, \theta\big) \\ \boldsymbol{\sigma}^{*[m]} &= \boldsymbol{\sigma}^{*[m]}\big(\boldsymbol{\sigma}^{*[j]};\ j = 0, 1, \dots, m-1, \boldsymbol{J}, \boldsymbol{J}_{[i]};\ i = 1, 2, \dots, n, \boldsymbol{g}, \theta\big) \\ \boldsymbol{q} &= \boldsymbol{q}\big(\boldsymbol{\sigma}^{*[j]};\ j = 0, 1, \dots, m-1, \boldsymbol{J}, \boldsymbol{J}_{[i]};\ i = 1, 2, \dots, n, \boldsymbol{g}, \theta\big) \end{aligned} \tag{4.12}$$

Thus the constitutive theories for these solids reduce to determination of $\boldsymbol{\sigma}^{*[m]}$ and $\boldsymbol{q}$.

4.2.1 Stress decomposition

To proceed further using (4.6) resulting from the entropy inequality, we perform decomposition of $\boldsymbol{\sigma}^*$ into equilibrium stress ${}_e\boldsymbol{\sigma}^*$ and deviatoric stress ${}_d\boldsymbol{\sigma}^*$:

$$\boldsymbol{\sigma}^* = {}_e\boldsymbol{\sigma}^* + {}_d\boldsymbol{\sigma}^* \tag{4.13}$$

in which we have

$$
{}_{e}\boldsymbol{\sigma}^{*} = {}_{e}\boldsymbol{\sigma}^{*}\big(\boldsymbol{J},\theta\big) \tag{4.14}
$$

$$
{}_{d}\boldsymbol{\sigma}^{*[m]} = {}_{d}\boldsymbol{\sigma}^{*[m]}\big({}_{d}\boldsymbol{\sigma}^{*[j]};\ j=0,1,\dots,m-1,\boldsymbol{J},\boldsymbol{J}_{[i]};\ i=1,2,\dots,n,\boldsymbol{g},\theta\big) \tag{4.15}
$$

In (4.15) we have changed dependence of ${}_{d}\boldsymbol{\sigma}^{*[m]}$ on ${}_{d}\boldsymbol{\sigma}^{*[j]};\ j=0,1,\dots,m-1$, as ${}_{e}\boldsymbol{\sigma}^{*}$ should play no role in the constitutive theory for ${}_{d}\boldsymbol{\sigma}^{*}$. Substituting (4.13) in (4.6), we obtain

$$
\left(\rho_0\frac{\partial\Phi}{\partial J_{ik}} - {}_{e}\sigma^{*}_{ki} - {}_{d}\sigma^{*}_{ki}\right)\dot{J}_{ik} + \frac{|J|q_i g_i}{\theta} \leq 0 \tag{4.16}
$$

or

$$
\left(\rho_0\frac{\partial\Phi}{\partial J_{ik}} - {}_{e}\sigma^{*}_{ki}\right)\dot{J}_{ik} - {}_{d}\sigma^{*}_{ki}\dot{J}_{ik} + \frac{|J|q_i g_i}{\theta} \leq 0 \tag{4.17}
$$

Since Φ is not a function of $\boldsymbol{J}_{[l]};\ l=1,2,\dots,n$, and neither is ${}_{e}\boldsymbol{\sigma}^{*}$ (in (4.14)), ${}_{e}\boldsymbol{\sigma}^{*}$ must be derivable from

$$
{}_{e}\sigma^{*}_{ki} = \rho_0\frac{\partial\Phi}{\partial J_{ik}} \text{ or } {}_{e}\boldsymbol{\sigma}^{*T} = \rho_0\frac{\partial\Phi}{\partial\boldsymbol{J}} \tag{4.18}
$$

Using (4.18), the inequality (4.17) reduces to

$$
-{}_{d}\sigma^{*}_{ki}\dot{J}_{ik} + \frac{|J|q_i g_i}{\theta} \leq 0 \tag{4.19}
$$

If we assume

$$
\frac{|J|q_i g_i}{\theta} \leq 0, \tag{4.20}
$$

then (4.19) is satisfied if

$$
{}_{d}\sigma^{*}_{ki}\dot{J}_{ik} > 0 \tag{4.21}
$$

Equation (4.21) requires that work expended due to deviatoric stress must be positive.

In view of (4.18), the stress decomposition (4.13) can be written as

$$\sigma^*_{ij} = \rho_0 \frac{\partial \Phi}{\partial J_{ji}} + {}_d\sigma^*_{ij} \tag{4.22}$$

Additionally, we have

$$\begin{aligned} {}_d\boldsymbol{\sigma}^{*[m]} &= {}_d\boldsymbol{\sigma}^{*[m]}\big({}_d\boldsymbol{\sigma}^{*[j]};\ j=0,1,\dots,m-1, \boldsymbol{J}, \boldsymbol{J}_{[l]};\ l=1,2,\dots,n, \boldsymbol{g}, \theta\big) && (4.23)\\ \Phi &= \Phi\big(\boldsymbol{J}, \theta\big) && (4.24)\\ \boldsymbol{q} &= \boldsymbol{q}\big({}_d\boldsymbol{\sigma}^{*[j]};\ j=0,1,\dots,m-1, \boldsymbol{J}, \boldsymbol{J}_{[l]};\ l=1,2,\dots,n, \boldsymbol{g}, \theta\big) && (4.25) \end{aligned}$$

Remarks. (1) When $\boldsymbol{J}_{[m]};\ m=1,2,\dots,n$ (all or any) are arguments of the dependent variables in the constitutive theories, we note that the entropy inequality requires decomposition of the stress tensor into equilibrium and deviatoric stress tensors.

(2) Based on the conditions (4.18) resulting from the entropy inequality (after stress decomposition), the equilibrium stress is deterministic from the Helmholtz free energy density, but the constitutive theory for deviatoric stress tensor is not. Thus in the case of rate constitutive theory, the entropy inequality (with stress decomposition) can only take us as far as (4.20) – (4.25). The constitutive theories for ${}_d\boldsymbol{\sigma}^*$ and $\boldsymbol{q}$ must satisfy (4.20) and (4.21) while ${}_e\boldsymbol{\sigma}^*$ must be derived using (4.18).

(3) The arguments of the dependent variables need further considerations: (i) first, due to frame invariance requirement, as $\boldsymbol{J}$ and $\boldsymbol{J}_{[l]};\ l=1,2,\dots,n$ are not frame invariant. (ii) Since the conditions resulting from the entropy inequality do not provide a mechanism for deriving constitutive theory for ${}_d\boldsymbol{\sigma}^*$, we shall consider theory of generators and invariants [1–3] for deriving constitutive theory for ${}_d\boldsymbol{\sigma}^*$, which requires arguments of ${}_d\boldsymbol{\sigma}^*$ to be tensors. ${}_d\boldsymbol{\sigma}^{*[j]};\ j=0,1,\dots,m$, $\boldsymbol{g}$ and θ are tensors of rank two, one, and zero, however $\boldsymbol{J}$ and $\boldsymbol{J}_{[i]};\ i=1,2,\dots,n$ are not tensors. Hence $\boldsymbol{J}$ and $\boldsymbol{J}_{[i]};\ i=1,2,\dots,n$ must be replaced by equivalent measures that are tensors.

4.3 Further considerations on the dependent variables and their arguments: axiom of frame invariance

Due to the principle of frame invariance and material objectivity, the rotation part of $\boldsymbol{J}$ and the rotation and rotation rates of $\boldsymbol{J}_{[m]};\ m = 1, 2, \ldots, n$ cannot be part of the constitutive theories. We consider the polar decomposition of $\boldsymbol{J}$

$$\boldsymbol{J} = \boldsymbol{R}\boldsymbol{S}_r = \boldsymbol{S}_l\boldsymbol{R} \tag{4.26}$$

In (4.26), $\boldsymbol{S}_r$ and $\boldsymbol{S}_l$ are right and left stretch tensors that are symmetric and positive definite, and $\boldsymbol{R}$ is the rotation matrix. Thus, $\boldsymbol{R}$ cannot be part of the constitutive theory. Hence, in (4.26), we must replace $\boldsymbol{J}$ by $\boldsymbol{S}_r$ (or $\boldsymbol{S}_l$ if so desired). However, $\boldsymbol{S}_r$ can be expressed in terms of Green's strain tensor $\boldsymbol{\varepsilon}$ or $\boldsymbol{\varepsilon}_{[0]}$ (material derivative of order zero)

$$\boldsymbol{S}_r^2 = \left(\boldsymbol{I} + 2\boldsymbol{\varepsilon}_{[0]}\right) \tag{4.27}$$

Thus, $\boldsymbol{S}_r$ can be replaced with $\boldsymbol{\varepsilon}_{[0]}$. Therefore, dependence on $\boldsymbol{J}$ can be replaced with $\boldsymbol{\varepsilon}_{[0]}$.

We also note that

$$\boldsymbol{J}_{[1]} = \boldsymbol{R}_{[1]}\boldsymbol{S}_r + \boldsymbol{R}(\boldsymbol{S}_r)_{[1]}, \tag{4.28}$$

but rotation $\boldsymbol{R}$ and rate of rotation $\boldsymbol{R}_{[1]}$ cannot be part of the constitutive theory, hence dependence on $\boldsymbol{J}_{[1]}$ can be replaced by dependence on $\boldsymbol{S}_r$ and $(\boldsymbol{S}_r)_{[1]}$. Thus, we have

$$\begin{aligned}
\boldsymbol{J} &= \boldsymbol{J}(\boldsymbol{S}_r) \text{ or } \boldsymbol{J}_{[0]} = \boldsymbol{J}_{[0]}(\boldsymbol{S}_r) \\
\boldsymbol{J}_{[1]} &= \boldsymbol{J}_{[1]}(\boldsymbol{S}_r, (\boldsymbol{S}_r)_{[1]}) \\
&\vdots \\
\boldsymbol{J}_{[n]} &= \boldsymbol{J}_{[n]}(\boldsymbol{S}_r, (\boldsymbol{S}_r)_{[1]}, \ldots (\boldsymbol{S}_r)_{[n]})
\end{aligned} \tag{4.29}$$

Using (4.29), $\boldsymbol{J}_{[k]};\ k = 0, 1, \ldots, n$ can be replaced by $(\boldsymbol{S}_r)_{[l]};\ l = 0, 1, \ldots, k$ and using

(4.27) we obtain

$$
\begin{aligned}
\boldsymbol{\varepsilon}_{[0]} &= \boldsymbol{\varepsilon}_{[0]}(\boldsymbol{S}_r) \\
\boldsymbol{\varepsilon}_{[1]} &= \boldsymbol{\varepsilon}_{[1]}(\boldsymbol{S}_r, (\boldsymbol{S}_r)_{[1]}) \\
&\vdots \\
\boldsymbol{\varepsilon}_{[n]} &= \boldsymbol{\varepsilon}_{[n]}(\boldsymbol{S}_r, (\boldsymbol{S}_r)_{[1]}, \dots (\boldsymbol{S}_r)_{[n]})
\end{aligned}
\tag{4.30}
$$

Using (4.30), $(\boldsymbol{S}_r)_{[k]};\ k = 0, 1, \dots, n$ can be replaced by $\boldsymbol{\varepsilon}_{[l]};\ l = 0, 1, \dots, n$. With Green's strain tensor and its material derivatives, the conjugate stress tensor is the second Piola-Kirchhoff stress tensor $\boldsymbol{\sigma}^{[0]}$ and its material derivatives. Hence, parallel to (4.22), we can write (using stress decomposition for $\boldsymbol{\sigma}^{[0]}$ similar to (4.13)):

$$\sigma_{ij}^{[0]} = {}_e\sigma_{ij}^{[0]} + {}_d\sigma_{ij}^{[0]} \tag{4.31}$$

$${}_d\boldsymbol{\sigma}^{[m]} = {}_d\boldsymbol{\sigma}^{[m]}\big(\boldsymbol{\varepsilon}_{[0]}, \boldsymbol{\varepsilon}_{[k]};\ k = 0, 1, \dots, n,\ {}_d\boldsymbol{\sigma}^{[l]};\ l = 0, 1, \dots, m-1, \boldsymbol{g}, \theta\big) \tag{4.32}$$

$$\boldsymbol{q} = \boldsymbol{q}\big(\boldsymbol{\varepsilon}_{[0]}, \boldsymbol{\varepsilon}_{[k]};\ k = 0, 1, \dots, n, \boldsymbol{\sigma}^{[l]};\ l = 0, 1, \dots, m-1, \boldsymbol{g}, \theta\big) \tag{4.33}$$

$$\Phi = \Phi\big(\boldsymbol{J}, \theta\big) \tag{4.34}$$

and

$${}_e\boldsymbol{\sigma}^{*T} = \rho_0 \frac{\partial \Phi\big(\boldsymbol{J}, \theta\big)}{\partial \boldsymbol{J}} \tag{4.35}$$

In (4.34), we cannot change the dependence of Φ on $\boldsymbol{J}$ by $\boldsymbol{\varepsilon}_{[0]}$, due to (4.35). This is necessary and intentional, as this form is useful in the derivation of ${}_e\boldsymbol{\sigma}^*$, from which ${}_e\bar{\boldsymbol{\sigma}}^{(0)}$ (equilibrium stress tensor based on contravariant Cauchy stress tensor) is derived, and then finally, ${}_e\boldsymbol{\sigma}^{[0]}$ is derived using ${}_e\bar{\boldsymbol{\sigma}}^{(0)}$.

Equations (4.32) and (4.33) define the desired choices of dependent variables and their argument tensors. We note that $\boldsymbol{\varepsilon}_{[k]};\ k = 0, 1, \dots, n$ and $\boldsymbol{\sigma}^{[l]};\ l = 0, 1, \dots, m$ are symmetric tensors of rank two, $\boldsymbol{g}$ is a tensor of rank one, and θ is a tensor of rank zero.

4.4 Constitutive theory for equilibrium stress ${}_e\boldsymbol{\sigma}^{[0]}$

Details of the derivations of the constitutive theories for ${}_e\boldsymbol{\sigma}^{[0]}$ for compressible as well as incompressible cases are presented in chapter 3 as well as reference [1]. Here we simply present the final expressions (for the sake of brevity).

4.4.1 Compressible matter: equilibrium stress tensor ${}_e\boldsymbol{\sigma}^{[0]}$

We consider the condition (4.35) and transform it using (4.36) into an equivalent form containing ${}_e\bar{\boldsymbol{\sigma}}^{(0)}$.

$$
{}_e\bar{\boldsymbol{\sigma}}^{(0)} = |J|^{-1}\,{}_e\boldsymbol{\sigma}^{*T}\boldsymbol{J}^T \tag{4.36}
$$

$$
{}_e\bar{\boldsymbol{\sigma}}^{(0)} = |J|^{-1}\rho_0\frac{\partial\bar{\Phi}(\bar{\rho},\bar{\theta})}{\partial\boldsymbol{J}}\boldsymbol{J}^T \tag{4.37}
$$

In (4.37), ${}_e\bar{\boldsymbol{\sigma}}^{(0)}$ is the equilibrium contravariant Cauchy stress tensor in Eulerian description. Following chapter 3 and reference [1], we can derive the following constitutive theory for ${}_e\bar{\boldsymbol{\sigma}}^{(0)}$ and ${}_e\boldsymbol{\sigma}^{(0)}$, the equilibrium contravariant Cauchy stress tensors in Eulerian and Lagrangian descriptions.

$$
{}_e\bar{\boldsymbol{\sigma}}^{(0)} = \bar{p}(\bar{\rho},\bar{\theta})\boldsymbol{I} \tag{4.38}
$$

$$
\bar{p}(\bar{\rho},\bar{\theta}) = -\bar{\rho}^2\frac{\partial\bar{\Phi}(\bar{\rho},\bar{\theta})}{\partial\bar{\rho}} \tag{4.39}
$$

and

$$
{}_e\boldsymbol{\sigma}^{(0)} = p(\rho,\theta)\boldsymbol{I} \tag{4.40}
$$

where

$$
p(\rho,\theta) = -\rho^2\frac{\partial\Phi(\rho,\theta)}{\partial\rho} \tag{4.41}
$$

As shown in chapter 3 and reference [1], from (4.40), it is straightforward to derive

$$ {}_{e}\boldsymbol{\sigma}^{[0]} = p(\rho,\theta)|\boldsymbol{J}|(\boldsymbol{J}^{T}\boldsymbol{J})^{-1} \tag{4.42}$$

$p(\rho,\theta)$ is thermodynamic pressure. If we assume compressive pressure to be positive, then $p(\rho,\theta)$ in (4.42) can be replaced by $-p(\rho,\theta)$. We also note that from continuity, $\rho_0 = |J|\rho$, hence ρ can be replaced by $\frac{\rho_0}{|J|}$, *i.e.*, $\rho(\boldsymbol{x},t)$ is deterministic if $\boldsymbol{J}$ is known in the current configuration.

4.4.2 Incompressible matter: equilibrium stress tensor ${}_{e}\sigma^{[0]}$

For incompressible matter, $\rho_0 = \rho$, hence $|J| = 1$, thus in this case, ${}_{e}\boldsymbol{\sigma}^{*}$ cannot be derived using (4.35) as in this case $\frac{\partial \Phi}{\partial \boldsymbol{J}} = 0$. However, when the incompressibility constraint is incorporated in the entropy inequality (4.1), then we can derive [1,62]

$$ {}_{e}\boldsymbol{\sigma}^{*T} = p(\theta)(\boldsymbol{J}^{T})^{-1} \tag{4.43}$$

which gives

$$ {}_{e}\bar{\boldsymbol{\sigma}}^{(0)} = \bar{p}(\bar{\theta})\boldsymbol{I} \text{ and } {}_{e}\boldsymbol{\sigma}^{(0)} = p(\theta)\boldsymbol{I} \tag{4.44}$$

From (4.44) it is straightforward [1,62] to derive

$$ {}_{e}\boldsymbol{\sigma}^{[0]} = p(\theta)(\boldsymbol{J}^{T}\boldsymbol{J})^{-1} \tag{4.45}$$

In this case also, if we assume compressive pressure to be positive, then $p(\theta)$ in (4.45) can be replaced by $-p(\theta)$.

4.5 Constitutive theory for deviatoric second Piola-Kirchhoff stress tensor ${}_{d}\boldsymbol{\sigma}^{[0]}$ and heat vector $\boldsymbol{q}$

Based on the derivation of the equilibrium stress, we note that it is derived from the pressure field. If we assume that the pressure field does not influence constitutive theory for deviatoric stress, then instead of $\boldsymbol{\sigma}^{[l]};\ l = 0,1,\ldots,m-1$ as arguments, we can use ${}_{d}\boldsymbol{\sigma}^{[l]};\ l = 0,1,\ldots,m-1$ as done in (4.15). We now have

$$ {}_{d}\boldsymbol{\sigma}^{[m]} = p|\boldsymbol{J}|(\boldsymbol{J}^T\boldsymbol{J})^{-1} + {}_{d}\boldsymbol{\sigma}^{[m]}\big(\boldsymbol{\varepsilon}_{[0]}, \boldsymbol{\varepsilon}_{[k]};\ k = 0,1,\ldots,n,\ {}_{d}\boldsymbol{\sigma}^{[l]};\ l = 0,1,\ldots,m-1, \boldsymbol{g}, \theta\big) \tag{4.46} $$

$$ \boldsymbol{q} = \boldsymbol{q}\big(\boldsymbol{\varepsilon}_{[0]}, \boldsymbol{\varepsilon}_{[k]};\ k = 0,1,\ldots,n,\ {}_{d}\boldsymbol{\sigma}^{[l]};\ l = 0,1,\ldots,m-1, \boldsymbol{g}, \theta\big) \tag{4.47} $$

We use theory of generators and invariants to derive constitutive theories of orders (m,n) for deviatoric second Piola-Kirchhoff stress tensor ${}_{d}\boldsymbol{\sigma}^{[0]}$ and heat vector $\boldsymbol{q}$.

4.5.1 Constitutive theory of orders (m,n) for the deviatoric second Piola-Kirchhoff stress tensor

In using the theory of generators and invariants to derive constitutive theory for deviatoric second Piola-Kirchhoff stress tensor, we consider (4.46) and express ${}_{d}\boldsymbol{\sigma}^{[m]}$ using a linear combination of $\boldsymbol{I}$ and the combined generators [31] of the argument tensors of ${}_{d}\boldsymbol{\sigma}^{[m]}$ and θ. The coefficients in the linear combination are functions of the combined invariants of the argument tensors of ${}_{d}\boldsymbol{\sigma}^{[m]}$. The material coefficients are derived by considering Taylor series expansions of the coefficients in the linear combination of the combined invariants and the temperature θ. ${}_{d}\boldsymbol{\sigma}^{[m]}$ is a symmetric tensor of rank two. $\boldsymbol{\varepsilon}_{[k]};\ k = 0,1,\ldots,n$ and ${}_{d}\boldsymbol{\sigma}^{[l]};\ l = 0,1,\ldots,m-1$ are also symmetric tensors of rank two, but $\boldsymbol{g}$ and $\boldsymbol{q}$ are tensors of rank one and zero. A complete list of generators and invariants can be found in Appendix A. Let ${}^{\sigma}\underset{\sim}{\boldsymbol{G}}^{i};\ i = 1,2,\ldots,N$ be the combined generators [31] that are symmetric tensors of rank two of the argument tensors $\boldsymbol{\varepsilon}_{[k]};\ k = 0,1,\dots,n$, ${}_d\boldsymbol{\sigma}^{[l]};\ l = 0,1,\dots,m-1$, and $\boldsymbol{g}$, and let ${}^{q\sigma}\underset{\sim}{I}^j;\ j = 1,2,\dots,M$ be the combined invariants of the same argument tensors. Then we can express ${}_d\boldsymbol{\sigma}^{[m]}$ in the current configuration using

$$ {}_d\boldsymbol{\sigma}^{[m]} = {}^{\sigma}\underset{\sim}{\alpha}^0 \boldsymbol{I} + \sum_{i=1}^{N} {}^{\sigma}\underset{\sim}{\alpha}^i \, {}^{\sigma}\underset{\sim}{\boldsymbol{G}}^i \tag{4.48} $$

in which

$$ {}^{\sigma}\underset{\sim}{\alpha}^i = {}^{\sigma}\underset{\sim}{\alpha}^i \left({}^{q\sigma}\underset{\sim}{I}^j;\ j = 1,2,\dots,M,\theta \right);\ i = 0,1,\dots,N \tag{4.49} $$

To determine material coefficients from ${}^{\sigma}\underset{\sim}{\alpha}^i;\ i = 0,1,\dots,N$ in (4.48), we consider a Taylor series expansion of coefficients ${}^{\sigma}\underset{\sim}{\alpha}^i;\ i = 0,1,\dots,N$ about a known configuration $\underline{\Omega}$ in ${}^{q\sigma}\underset{\sim}{I}^j;\ j = 1,2,\dots,M$ and θ and retain only up to linear terms in the invariants and temperature θ.

$$ {}^{\sigma}\underset{\sim}{\alpha}^i = {}^{\sigma}\underset{\sim}{\alpha}^i\big|_{\underline{\Omega}} + \sum_{j=1}^{M} \left. \frac{\partial \, {}^{\sigma}\underset{\sim}{\alpha}^i}{\partial \, {}^{q\sigma}\underset{\sim}{I}^j} \right|_{\underline{\Omega}} \left({}^{q\sigma}\underset{\sim}{I}^j - ({}^{q\sigma}\underset{\sim}{I}^j)_{\underline{\Omega}} \right) + \left. \frac{\partial \, {}^{\sigma}\underset{\sim}{\alpha}^i}{\partial \theta} \right|_{\underline{\Omega}} \left(\theta - \theta_{\underline{\Omega}} \right);\ i = 0,1,\dots,N \tag{4.50} $$

When (4.50) is substituted in (4.48), we obtain the final form of the most general rate constitutive theory of orders (m,n) for the deviatoric second Piola-Kirchhoff stress tensor. Details of the resulting material coefficients are given in the following.

$$ \begin{aligned} {}_d\boldsymbol{\sigma}^{[m]} = & \left({}^{\sigma}\underset{\sim}{\alpha}^0\big|_{\underline{\Omega}} + \sum_{j=1}^{M} \left. \frac{\partial \, {}^{\sigma}\underset{\sim}{\alpha}^0}{\partial \, {}^{q\sigma}\underset{\sim}{I}^j} \right|_{\underline{\Omega}} \left({}^{q\sigma}\underset{\sim}{I}^j - ({}^{q\sigma}\underset{\sim}{I}^j)_{\underline{\Omega}} \right) + \left. \frac{\partial \, {}^{\sigma}\underset{\sim}{\alpha}^0}{\partial \theta} \right|_{\underline{\Omega}} \left(\theta - \theta_{\underline{\Omega}} \right) \right) \boldsymbol{I} \\ & + \sum_{i=1}^{N} \left({}^{\sigma}\underset{\sim}{\alpha}^i\big|_{\underline{\Omega}} + \sum_{j=1}^{M} \left. \frac{\partial \, {}^{\sigma}\underset{\sim}{\alpha}^i}{\partial \, {}^{q\sigma}\underset{\sim}{I}^j} \right|_{\underline{\Omega}} \left({}^{q\sigma}\underset{\sim}{I}^j - ({}^{q\sigma}\underset{\sim}{I}^j)_{\underline{\Omega}} \right) + \left. \frac{\partial \, {}^{\sigma}\underset{\sim}{\alpha}^i}{\partial \theta} \right|_{\underline{\Omega}} \left(\theta - \theta_{\underline{\Omega}} \right) \right) {}^{\sigma}\underset{\sim}{\boldsymbol{G}}^i \end{aligned} \tag{4.51} $$

Collecting coefficients (those defined in the known configuration $\underline{\Omega}$) of $\boldsymbol{I}$, ${}^{q\sigma}\underset{\sim}{I}^j \boldsymbol{I}$, ${}^{\sigma}\underset{\sim}{\boldsymbol{G}}^i$,

${}^{q\sigma}\underset{\sim}{\underline{I}}^j\,{}^{\sigma}\underset{\sim}{\boldsymbol{G}}^i$, $(\theta-\theta_{\underline{\Omega}})\,{}^{\sigma}\underset{\sim}{\boldsymbol{G}}^i$, and $(\theta-\theta_{\underline{\Omega}})\boldsymbol{I}$ in (4.51) and defining

$$
\begin{aligned}
&\underline{\sigma}^0|_{\underline{\Omega}} = {}^{\sigma}\underset{\sim}{\underline{\alpha}}^0|_{\underline{\Omega}} + \sum_{j=1}^{M} \left.\frac{\partial\, {}^{\sigma}\underset{\sim}{\underline{\alpha}}^0}{\partial\, {}^{q\sigma}\underset{\sim}{\underline{I}}^j}\right|_{\underline{\Omega}} ({}^{q\sigma}\underset{\sim}{\underline{I}}^j)_{\underline{\Omega}} && {}^{\sigma}\underline{a}_j = \left.\frac{\partial\, {}^{\sigma}\underset{\sim}{\underline{\alpha}}^0}{\partial\, {}^{q\sigma}\underset{\sim}{\underline{I}}^j}\right|_{\underline{\Omega}} ;\; j=1,2,\dots,M \\
&{}^{\sigma}\underline{b}_i = {}^{\sigma}\underset{\sim}{\underline{\alpha}}^i|_{\underline{\Omega}} + \sum_{j=1}^{M} \left.\frac{\partial\, {}^{\sigma}\underset{\sim}{\underline{\alpha}}^i}{\partial\, {}^{q\sigma}\underset{\sim}{\underline{I}}^j}\right|_{\underline{\Omega}} ;\; i=1,2,\dots,N && {}^{\sigma}\underline{c}_{ij} = \left.\frac{\partial\, {}^{\sigma}\underset{\sim}{\underline{\alpha}}^i}{\partial\, {}^{q\sigma}\underset{\sim}{\underline{I}}^j}\right|_{\underline{\Omega}} ;\; \begin{array}{l} i=1,2,\dots,N \\ j=1,2,\dots,M \end{array} \\
&{}^{\sigma}\underline{d}_i = -\left.\frac{\partial\, {}^{\sigma}\underset{\sim}{\underline{\alpha}}^i}{\partial\theta}\right|_{\underline{\Omega}} ;\; i=1,2,\dots,N && (\underline{\alpha}_{tm})_{\underline{\Omega}} = -\left.\frac{\partial\, {}^{\sigma}\underset{\sim}{\underline{\alpha}}^0}{\partial\theta}\right|_{\underline{\Omega}}
\end{aligned}
\tag{4.52}
$$

Equation (4.51) can be written as

$$
\begin{aligned}
{}_d\boldsymbol{\sigma}^{[m]} = {} & \underline{\sigma}^0_{\underline{\Omega}}\boldsymbol{I} + \sum_{j=1}^{M} {}^{\sigma}\underline{a}_j\, {}^{q\sigma}\underset{\sim}{\underline{I}}^j\boldsymbol{I} + \sum_{i=1}^{N} {}^{\sigma}\underline{b}_i\, {}^{\sigma}\underset{\sim}{\boldsymbol{G}}^i \\
& + \sum_{i=1}^{N}\sum_{j=1}^{M} {}^{\sigma}\underline{c}_{ij}\, {}^{q\sigma}\underset{\sim}{\underline{I}}^j\, {}^{\sigma}\underset{\sim}{\boldsymbol{G}}^i + \sum_{i=1}^{N} {}^{\sigma}\underline{d}_i(\theta-\theta_{\underline{\Omega}})\, {}^{\sigma}\underset{\sim}{\boldsymbol{G}}^i \\
& - (\underline{\alpha}_{tm})_{\underline{\Omega}}(\theta-\theta_{\underline{\Omega}})\boldsymbol{I}
\end{aligned}
\tag{4.53}
$$

$\underline{\sigma}^0|_{\underline{\Omega}}$ is the initial stress in the known configuration $\underline{\Omega}$. This constitutive theory for ${}_d\boldsymbol{\sigma}^{[0]}$ requires $(M+N+MN+N+1)$ material coefficients. The material coefficients defined in (4.52) are functions of $\theta_{\underline{\Omega}}$ and $({}^{q\sigma}\underset{\sim}{\underline{I}}^j)_{\underline{\Omega}};\ j=1,2,\dots,M$ in the known configuration $\underline{\Omega}$. This constitutive theory is based on integrity and hence is complete.

4.5.2 Constitutive theory of orders (m,n) for $\boldsymbol{q}$

In this derivation we consider the argument tensors of $\boldsymbol{q}$ in (4.47). These are in agreement with those for ${}_d\boldsymbol{\sigma}^{[m]}$, hence this constitutive theory is consistent with the rate constitutive theory for deviatoric second Piola-Kirchhoff stress tensor in section 4.5.1. We express $\boldsymbol{q}$ as a linear combination of the combined generators of the argument tensors of $\boldsymbol{q}$ and θ. As in the case of ${}_d\boldsymbol{\sigma}^{[m]}$, here also the coefficients in the linear combination are functions of the combined invariants of the argument tensors of $\boldsymbol{q}$. The material coefficients in this case are also derived using a Taylor series expansion of each coefficient in the linear combination about a known configuration $\underline{\Omega}$. Since $\boldsymbol{q}$ is a tensor of rank one, the combined generators of the argument tensors of $\boldsymbol{q}$ must also be tensors of rank one. We keep in mind that $\boldsymbol{\varepsilon}_{[k]};\ k = 0,1,\dots,n$ and ${}_d\boldsymbol{\sigma}^{[l]};\ l = 0,1,\dots,m$ are symmetric tensors of rank two, while $\boldsymbol{g}$, θ are tensors of rank one and zero respectively. Let ${}^q\underset{\sim}{\boldsymbol{G}}^i;\ i = 1,2,\dots,\widetilde{N}$ be the combined generators of the argument tensors of $\boldsymbol{q}$ that are tensors of rank one and let ${}^{q\sigma}\underset{\sim}{I}^j;\ j = 1,2,\dots,M$ be the combined invariants of the same argument tensors of $\boldsymbol{q}$, which are the same as those for ${}_d\boldsymbol{\sigma}^{[m]}$. Then, we can express $\boldsymbol{q}$ in the current configuration as a linear combination of ${}^q\underset{\sim}{\boldsymbol{G}}^i;\ i = 1,2,\dots,\widetilde{N}$:

$$\boldsymbol{q} = -\sum_{i=1}^{\widetilde{N}} {}^q\underset{\sim}{\alpha}^i\, {}^q\underset{\sim}{\boldsymbol{G}}^i \tag{4.54}$$

The reasons for the absence of constant terms in (4.54) and the negative sign are well known [1–3,58,64].

$${}^q\underset{\sim}{\alpha}^i = {}^q\underset{\sim}{\alpha}^i\left({}^{q\sigma}\underset{\sim}{I}^j;\ j = 1,2,\dots,M,\theta\right) \tag{4.55}$$

To determine the material coefficients from ${}^q\underset{\sim}{\alpha}^i$ in (4.54), we consider a Taylor series expansion of each ${}^q\underset{\sim}{\alpha}^i$ about a known configuration $\underline{\Omega}$ in invariants ${}^{q\sigma}\underset{\sim}{I}^j$ and temperature θ and retain only up to linear terms in ${}^{q\sigma}\underset{\sim}{I}^j;\ j = 1,2,\dots,M$ and θ.

$${}^q\underset{\sim}{\alpha}^i = {}^q\underset{\sim}{\alpha}^i\big|_{\underline{\Omega}} + \sum_{j=1}^{M} \left.\frac{\partial\, {}^q\underset{\sim}{\alpha}^i}{\partial\, {}^{q\sigma}\underset{\sim}{I}^j}\right|_{\underline{\Omega}} \left({}^{q\sigma}\underset{\sim}{I}^j - ({}^{q\sigma}\underset{\sim}{I}^j)_{\underline{\Omega}}\right) + \left.\frac{\partial\, {}^q\underset{\sim}{\alpha}^i}{\partial\theta}\right|_{\underline{\Omega}} \left(\theta - \theta_{\underline{\Omega}}\right);\ i = 1,2,\dots,\widetilde{N} \tag{4.56}$$

When we substitute (4.56) into (4.54), we obtain the most general constitutive theory of orders (m,n) for $\boldsymbol{q}$. Details of the derivation and the material coefficients are given in the following.

Substituting from (4.56) into (4.54)

$$
\boldsymbol{q} = -\sum_{i=1}^{\widetilde{N}} \left({}^{q}\underset{\sim}{\alpha}^{i}|_{\underline{\Omega}} + \sum_{j=1}^{M} \left. \frac{\partial\, {}^{q}\underset{\sim}{\alpha}^{i}}{\partial\, {}^{q\sigma}\underset{\sim}{I}^{j}} \right|_{\underline{\Omega}} \left({}^{q\sigma}\underset{\sim}{I}^{j} - ({}^{q\sigma}\underset{\sim}{I}^{j})_{\underline{\Omega}} \right) + \left. \frac{\partial\, {}^{q}\underset{\sim}{\alpha}^{i}}{\partial \theta} \right|_{\underline{\Omega}} (\theta - \theta_{\underline{\Omega}}) \right) {}^{q}\underset{\sim}{\boldsymbol{G}}^{i} \tag{4.57}
$$

Collecting coefficients (only those defined in the known configuration $\underline{\Omega}$) of ${}^{q}\underset{\sim}{\boldsymbol{G}}^{i}$, ${}^{q\sigma}\underset{\sim}{I}^{j}\,{}^{q}\underset{\sim}{\boldsymbol{G}}^{i}$, and $(\theta - \theta_{\underline{\Omega}})\,{}^{q}\underset{\sim}{\boldsymbol{G}}^{i}$ in (4.57) and defining the material coefficients

$$
\begin{aligned}
{}^{q}\underline{b}_{i} &= {}^{q}\underset{\sim}{\alpha}^{i}|_{\underline{\Omega}} + \sum_{j=1}^{M} \left. \frac{\partial\, {}^{q}\underset{\sim}{\alpha}^{i}}{\partial\, {}^{q\sigma}\underset{\sim}{I}^{j}} \right|_{\underline{\Omega}} \; ; \; i = 1, 2, \dots, \widetilde{N} \\
{}^{q}\underline{c}_{ij} &= \left. \frac{\partial\, {}^{q}\underset{\sim}{\alpha}^{i}}{\partial\, {}^{q\sigma}\underset{\sim}{I}^{j}} \right|_{\underline{\Omega}} \; ; \; i = 1, 2, \dots, \widetilde{N}; \; j = 1, 2, \dots, M \\
{}^{q}\underline{d}_{i} &= \left. \frac{\partial\, {}^{q}\underset{\sim}{\alpha}^{i}}{\partial \theta} \right|_{\underline{\Omega}} \; ; \; i = 1, 2, \dots, \widetilde{N}
\end{aligned} \tag{4.58}
$$

Equation (4.57) can be written as

$$
\boldsymbol{q} = -\sum_{i=1}^{\widetilde{N}} {}^{q}\underline{b}_{i}\, {}^{q}\underset{\sim}{\boldsymbol{G}}^{i} - \sum_{i=1}^{\widetilde{N}} \sum_{j=1}^{M} {}^{q}\underline{c}_{ij}\, {}^{q\sigma}\underset{\sim}{I}^{j}\, {}^{q}\underset{\sim}{\boldsymbol{G}}^{i} - \sum_{i=1}^{\widetilde{N}} {}^{q}\underline{d}_{i} (\theta - \theta_{\underline{\Omega}})\, {}^{q}\underset{\sim}{\boldsymbol{G}}^{i} \tag{4.59}
$$

This constitutive theory for $\boldsymbol{q}$ requires $(\widetilde{N} + \widetilde{N}M + \widetilde{N})$ material coefficients defined in (4.58). The material coefficients defined in (4.58) are functions of invariants ${}^{q\sigma}\underset{\sim}{I}^{j}|_{\underline{\Omega}}$; $j = 1, 2, \dots, M$ and temperature $\theta|_{\underline{\Omega}}$ in the known configuration $\underline{\Omega}$. This constitutive theory is also based on integrity, hence is complete.

Remarks. (1) Using the general derivations presented in sections 4.5.1 and 4.5.2 for rate theories of orders (m, n) for the deviatoric second Piola-Kirchhoff stress tensor and the heat vector, the constitutive theories of any desired order can be obtained.

(2) The rate constitutive theories for deviatoric second Piola-Kirchhoff stress tensor and heat vector are consistent with each other as they use the same argument tensors. The constitutive theory for $\boldsymbol{q}$ demonstrates the influence of $\boldsymbol{\varepsilon}_{[k]};\ k = 0, 1, \dots, n$ and ${}_d\boldsymbol{\sigma}^{[l]};\ l = 0, 1, \dots, m-1$ on $\boldsymbol{q}$ as well as their interaction with temperature gradient $\boldsymbol{g}$. In a later section we consider much simplified constitutive theory for $\boldsymbol{q}$ to demonstrate this point more clearly.

4.5.3 Simplified rate constitutive theories of orders one ($m = 1$) and n for deviatoric second Piola-Kirchhoff stress tensor

For the rate constitutive theories of orders $(m = 1, n)$, we have

$$
{}_d\boldsymbol{\sigma}^{[1]} = {}_d\boldsymbol{\sigma}^{[1]}\big({}_d\boldsymbol{\sigma}^{[0]}, \boldsymbol{\varepsilon}_{[i]};\ i = 0, 1, \dots, n, \boldsymbol{g}, \theta\big)
$$

Consider simplified constitutive theories based on the following assumptions:

(1) The constitutive theories are linear in the components of the argument tensors.

(2) We neglect all product terms (in the current configuration) related to the argument tensors.

Based on these assumptions, we only have ${}_d\boldsymbol{\sigma}^{[0]}$, $\boldsymbol{\varepsilon}_{[i]};\ i = 0, 1, \dots, n$ as generators, and the only invariants to be considered are $\operatorname{tr} {}_d\boldsymbol{\sigma}^{[0]}$ and $\operatorname{tr} \boldsymbol{\varepsilon}_{[i]};\ i = 0, 1, \dots, n$. The resulting simplified rate theories of order $(1, n)$ are given by

$$
\begin{aligned}
{}_d\boldsymbol{\sigma}^{[1]} &= \underline{\boldsymbol{\sigma}}^0\big|_{\underline{\Omega}} - c_1\, {}_d\boldsymbol{\sigma}^{[0]} - c_2\big(\operatorname{tr}\, {}_d\boldsymbol{\sigma}^{[0]}\big)\boldsymbol{I} \\
&\quad + \sum_{i=0}^{n} a_i^1 \boldsymbol{\varepsilon}_{[i]} + \sum_{i=0}^{n} a_i^2\big(\operatorname{tr} \boldsymbol{\varepsilon}_{[i]}\big)\boldsymbol{I} - \underline{\alpha}_{tm}(\theta - \theta_{\underline{\Omega}})\boldsymbol{I}.
\end{aligned}
\tag{4.60}
$$

The material coefficients c_1, c_2, a_i^1, a_i^2, and $\underline{\alpha}_{tm}$ are functions of the invariants and temperature θ in the known configuration $\underline{\Omega}$. We can write (4.60) in matrix and vector form using Voigt's notation (in the absence of the first and last terms in (4.60) without loss of generality)

$$\{\, {}_d\sigma^{[1]}\} + [\underset{\sim}{c}]\{\, {}_d\sigma^{[0]}\} = [\underset{\sim}{a}_0]\{(\varepsilon_{[0]})\} + \sum_{i=1}^{n} [\underset{\sim}{a}_i]\{(\varepsilon_{[i]})\} \tag{4.61}$$

in which

$$\{\, {}_d\sigma^{[1]}\}^T = [\, {}_d\sigma^{[1]}_{x_1x_1},\, {}_d\sigma^{[1]}_{x_2x_2},\, {}_d\sigma^{[1]}_{x_3x_3},\, {}_d\sigma^{[1]}_{x_2x_3},\, {}_d\sigma^{[1]}_{x_3x_1},\, {}_d\sigma^{[1]}_{x_1x_2}] \tag{4.62}$$

and

$$[\underset{\sim}{c}] = \begin{bmatrix} c_1+c_2 & c_2 & c_2 & 0 & 0 & 0 \\ c_2 & c_1+c_2 & c_2 & 0 & 0 & 0 \\ c_2 & c_2 & c_1+c_2 & 0 & 0 & 0 \\ 0 & 0 & 0 & c_1 & 0 & 0 \\ 0 & 0 & 0 & 0 & c_1 & 0 \\ 0 & 0 & 0 & 0 & 0 & c_1 \end{bmatrix} \tag{4.63}$$

Similarly, components of $[\underset{\sim}{a}_i];\ i = 0,1,\dots,n$ can be obtained using (4.63) by replacing c_1 and c_2 with a_i^1 and a_i^2. The components of ${}_d\sigma^{[0]}$ and $\varepsilon_{[i]};\ i = 0,1,\dots,n$ in $\{\, {}_d\sigma^{[0]}\}$ and $\{(\varepsilon_{[i]})\};\ i = 0,1,\dots,n$ are arranged in the same fashion as those of ${}_d\sigma^{[1]}$ in (4.62).

Remarks. (1) The coefficients of the matrix $[\underset{\sim}{a}_0]$ are completely defined using material coefficients a_0^1 and a_0^2. For linear elastic solids, $(\underset{\sim}{a}_0)_{ij}$ of $[\underset{\sim}{a}_0]$ are functions of the modulus of elasticity and Poisson's ratio, or Lamé's constants.

(2) The dissipation mechanism is due to $[\underset{\sim}{a}_i]\{(\varepsilon_{[i]})\};\ i = 1,2,\dots,n$ terms in (4.61). Each $\varepsilon_{[i]};\ i = 1,2,\dots,n$ requires two material coefficients.

(3) Using (4.61), various order rate theories can be obtained by choosing appropriate values of n. For all these theories, m remains one.

(4) Equation (4.61) is a first-order partial differential equation in ${}_d\sigma^{[0]}$, hence it can be integrated in time to obtain an integral expression for ${}_d\sigma^{[0]}$. The integrand in this expression is an exponentially decaying function and is called the memory modulus. Such materials, upon cessation of a disturbance, require a finite amount of time to

achieve a relaxed or stress-free state (stress relaxation). Such materials are referred to as materials with fading memory as these materials only have recollection of immediately preceding events but not those in the distant past.

(5) It is important to reiterate that (4.61) holds for finite deformation as the conjugate stress and strain measures are for finite deformation.

4.5.3.1 *Memory modulus or relaxation modulus*

In order to derive the expression for memory modulus, it is perhaps simpler to consider a one-dimensional case (in the x_1-coordinate) of (4.61) and to denote ${}_{d}\sigma^{[0]}_{x_1x_1}$ as ${}_{d}\sigma_{x_1x_1}$, ${}_{d}\sigma^{[1]}_{x_1x_1}$ as $\frac{\partial}{\partial t}({}_{d}\sigma_{x_1x_1})$, and $\varepsilon_{[i]}$ as $\frac{\partial^i \varepsilon}{\partial t^i}$; $i = 0, 1, \ldots, n$. Then, for the 1-d case, (4.61) can be written as

$$\frac{\partial}{\partial t}({}_{d}\sigma_{x_1x_1}) + \underset{\sim}{c}_{11}\, {}_{d}\sigma_{x_1x_!} = (\underset{\sim}{a}_0)_{11}(\varepsilon_{[0]})_{x_1x_1} + \sum_{i=1}^{n}(\underset{\sim}{a}_i)_{11}(\varepsilon_{[i]})_{x_1x_1} \tag{4.64}$$

Divide through by $\underset{\sim}{c}_{11}$ (as $\underset{\sim}{c}_{11}$ is always > 0) and define

$$\lambda = \frac{1}{\underset{\sim}{c}_{11}}; \quad n_i = \frac{(\underset{\sim}{a}_i)_{11}}{\underset{\sim}{c}_{11}}; \quad e_1 = \frac{(\underset{\sim}{a}_0)_{11}}{\underset{\sim}{c}_{11}} \tag{4.65}$$

Then (4.64) can be written as

$${}_{d}\sigma_{x_1x_1} + \lambda\frac{\partial({}_{d}\sigma_{x_1x_1})}{\partial t} = e_1(\varepsilon)_{x_1x_1} + \sum_{i=1}^{n} n_i \frac{\partial^i(\varepsilon)_{x_1x_1}}{\partial t^i} \tag{4.66}$$

We recall that the differential equation

$$\frac{dy}{dt} + P(t)y = Q(t) \tag{4.67}$$

has a solution

$$y = e^{-\int P(t)dt}\left[\int Q(t)e^{\int P(t)dt}dt + C\right] \tag{4.68}$$

where C is the constant of integration.

Comparing (4.67) with (4.66), we have

$$P(t) = \frac{1}{\lambda}, \quad Q(t) = \frac{1}{\lambda}\left(e_1(\varepsilon)_{x_1x_1} + \sum_{i=1}^{n} n_i \frac{\partial^i(\varepsilon)_{x_1x_1}}{\partial t^i}\right) \tag{4.69}$$

Using (4.69) in (4.68), we obtain

$${}_d\sigma_{x_1x_1} = e^{-\frac{t}{\lambda}}\left[\int Q(t')e^{\frac{t'}{\lambda}}dt' + C\right] \tag{4.70}$$

Using the integration limits $-\infty$ to t

$${}_d\sigma_{x_1x_1} = \int_{-\infty}^{t} Q(t')e^{\frac{-(t-t')}{\lambda}}dt' + Ce^{-\frac{t}{\lambda}} \tag{4.71}$$

If $Q(t)$ is finite at $t = -\infty$, then ${}_d\sigma_{x_1x_1}$ is also finite, hence the constant C must be zero. Thus, we have

$${}_d\sigma_{x_1x_1} = \int_{-\infty}^{t}\left(Q(t')e^{\frac{-(t-t')}{\lambda}}\right)dt' \tag{4.72}$$

The quantity in the brackets is called the relaxation modulus or memory modulus. Based on (4.72), the stress at time t (current configuration) depends upon strain and strain rates (due to $Q(\cdot)$) at time t, as well as the strain and strain rates at all past times t' with a weighting factor (the relaxation modulus) that decays exponentially as one goes backward in time. Thus, such materials have fading memory or stress relaxation.

When λ is zero in (4.66), then we have

$${}_d\sigma_{x_1x_1} = e_1(\varepsilon)_{x_1x_1} + \sum_{i=1}^{n} n_i \frac{\partial^i(\varepsilon)_{x_1x_1}}{\partial t^i} \tag{4.73}$$

The materials described by (4.73) have no stress relaxation or fading memory, due to the nonexistence of a relaxation modulus. Equation (4.73) is the same as that derived in chapter 3 for thermoviscoelastic solids without memory.

4.5.3.2 *Simplified 1-d case for incompressible medium with infinitesimal deformation: rate theories of orders* $(1,n)$ *and* $(1,1)$

For this case, there is no distinction between second Piola-Kirchhoff stress and Cauchy stress, and

$$(\varepsilon)_{x_1x_1} = \frac{\partial u_{x_1}}{\partial x_1}; \quad (\varepsilon_{[i]})_{x_1x_1} = \frac{\partial^i}{\partial t^i}\left(\frac{\partial u_{x_1}}{\partial x_1}\right) \tag{4.74}$$

Hence, (4.66) reduces to

$$_d\sigma_{x_1x_1} + \lambda\frac{\partial(\,_d\sigma_{x_1x_1})}{\partial t} = e_1\frac{\partial u_{x_1}}{\partial x_1} + \sum_{i=1}^{n} n_i\frac{\partial^i}{\partial t^i}\left(\frac{\partial u_{x_1}}{\partial x_1}\right) \tag{4.75}$$

When $(m,n) = (1,1)$, (4.75) reduces to

$$_d\sigma_{x_1x_1} + \lambda\frac{\partial(\,_d\sigma_{x_1x_1})}{\partial t} = e_1\frac{\partial u_{x_1}}{\partial x_1} + n_1\frac{\partial}{\partial t}\left(\frac{\partial u_{x_1}}{\partial x_1}\right) \tag{4.76}$$

This is similar to the Zener model [50] but differs due to the fact that (4.76) uses $_d\sigma_{x_1x_1}$, whereas the Zener model uses $\sigma_{x_1x_1}$. Based on the derivations presented in this chapter, the Zener model does not have a continuum mechanics basis.

4.5.4 Simplified rate constitutive theories of orders $(1,n)$ and $(1,1)$ for heat vector $\boldsymbol{q}$

In this section, we consider a simplified form of the rate constitutive theories of orders n and one for the heat vector $\boldsymbol{q}$, in which $\boldsymbol{q}$ is a linear function of the components of $_d\boldsymbol{\sigma}^{[0]}$ and $\boldsymbol{\varepsilon}_{[i]};\ i = 0,1,\ldots,n$, but the terms containing products of the components of those terms are neglected. All nonlinear terms in $\boldsymbol{g}$ are retained.

Thus, in this theory, we only consider the following generators:

$$\boldsymbol{g}, \quad _d\boldsymbol{\sigma}^{[0]}\boldsymbol{g}, \quad \boldsymbol{\varepsilon}_{[i]}\boldsymbol{g};\ i = 0,1,\ldots,n \tag{4.77}$$

and the invariants

$$\begin{aligned} &\boldsymbol{g}\cdot\boldsymbol{g}, \quad \operatorname{tr}\, {}_{d}\boldsymbol{\sigma}^{[0]}, \quad \operatorname{tr}\boldsymbol{\varepsilon}_{[i]};\ i=0,1,\dots,n \\ &\boldsymbol{g}\cdot({}_{d}\boldsymbol{\sigma}^{[0]}\boldsymbol{g}), \quad \boldsymbol{g}\cdot(\boldsymbol{\varepsilon}_{[i]}\boldsymbol{g});\ i=0,1,\dots,n \end{aligned} \tag{4.78}$$

in the current configuration after a Taylor series expansion of the coefficients ${}^{q}\alpha^{i}$ about a known configuration $\underline{\Omega}$, keeping in mind that this theory is linear in the components of the argument tensors of $\boldsymbol{q}$ but nonlinear in $\boldsymbol{g}$.

$$\begin{aligned} \boldsymbol{q} = &-\widetilde{a}\boldsymbol{g} - \widetilde{b}\,{}_{d}\boldsymbol{\sigma}^{[0]}\boldsymbol{g} - \sum_{i=0}^{n}\widetilde{c}_i\boldsymbol{\varepsilon}_{[i]}\boldsymbol{g} \\ &-\underset{\sim}{a}(\boldsymbol{g}\cdot\boldsymbol{g})\boldsymbol{g} - \underset{\sim}{b}\big(\operatorname{tr}\,{}_{d}\boldsymbol{\sigma}^{[0]}\big)\boldsymbol{g} - \sum_{i=0}^{n}\underset{\sim}{c}_i\big(\operatorname{tr}\boldsymbol{\varepsilon}_{[i]}\big)\boldsymbol{g} \\ &-\underset{\sim}{\alpha}\big(\boldsymbol{g}\cdot({}_{d}\boldsymbol{\sigma}^{[0]}\boldsymbol{g})\big)\boldsymbol{g} - \sum_{i=0}^{n}\underset{\sim}{e}_i\big(\boldsymbol{g}\cdot(\boldsymbol{\varepsilon}_{[i]}\boldsymbol{g})\big)\boldsymbol{g} \\ &-\underset{\sim}{f}(\theta-\theta_{\underline{\Omega}})\boldsymbol{g} \end{aligned} \tag{4.79}$$

Using (4.79), simplified rate constitutive theory of any order can be obtained explicitly, for example, when $n=1$, (4.79) reduces to the following:

$$\begin{aligned} \boldsymbol{q} = &-\widetilde{a}\boldsymbol{g} - \widetilde{b}\,{}_{d}\boldsymbol{\sigma}^{[0]}\boldsymbol{g} - \widetilde{c}_0\boldsymbol{\varepsilon}_{[0]}\boldsymbol{g} - \widetilde{c}_1\boldsymbol{\varepsilon}_{[1]}\boldsymbol{g} \\ &-\underset{\sim}{a}(\boldsymbol{g}\cdot\boldsymbol{g})\boldsymbol{g} - \underset{\sim}{b}\big(\operatorname{tr}\,{}_{d}\boldsymbol{\sigma}^{[0]}\big)\boldsymbol{g} - \underset{\sim}{c}_0\big(\operatorname{tr}\boldsymbol{\varepsilon}_{[0]}\big)\boldsymbol{g} - \underset{\sim}{c}_1\big(\operatorname{tr}\boldsymbol{\varepsilon}_{[1]}\big)\boldsymbol{g} \\ &-\underset{\sim}{d}\big(\boldsymbol{g}\cdot({}_{d}\boldsymbol{\sigma}^{[0]}\boldsymbol{g})\big)\boldsymbol{g} - \underset{\sim}{e}_0\big(\boldsymbol{g}\cdot(\boldsymbol{\varepsilon}_{[0]}\boldsymbol{g})\big)\boldsymbol{g} - \underset{\sim}{e}_1\big(\boldsymbol{g}\cdot(\boldsymbol{\varepsilon}_{[1]}\boldsymbol{g})\big)\boldsymbol{g} \\ &-\underset{\sim}{f}(\theta-\theta_{\underline{\Omega}})\boldsymbol{g} \end{aligned} \tag{4.80}$$

It is perhaps more instructive to rearrange the terms in (4.80) in the following form:

$$\begin{aligned} \boldsymbol{q} = -\bigg(&\Big(\widetilde{a} + \underset{\sim}{a}\boldsymbol{g}\cdot\boldsymbol{g} + \underset{\sim}{b}\big(\operatorname{tr}\,{}_{d}\boldsymbol{\sigma}^{[0]}\big) + \underset{\sim}{c}_0\big(\operatorname{tr}\boldsymbol{\varepsilon}_{[0]}\big) + \underset{\sim}{c}_1\big(\operatorname{tr}\boldsymbol{\varepsilon}_{[1]}\big)\Big)\boldsymbol{I} \\ &+ \widetilde{b}\,{}_{d}\boldsymbol{\sigma}^{[0]} + \widetilde{c}_0\boldsymbol{\varepsilon}_{[0]} + \widetilde{c}_1\boldsymbol{\varepsilon}_{[1]} \\ &+ \Big(\boldsymbol{g}\cdot\big(\underset{\sim}{d}\,{}_{d}\boldsymbol{\sigma}^{[0]} + \underset{\sim}{e}_0\boldsymbol{\varepsilon}_{[0]} + \underset{\sim}{e}_1\boldsymbol{\varepsilon}_{[1]}\big)\boldsymbol{g}\Big)\boldsymbol{I}\bigg)\boldsymbol{g} - \underset{\sim}{f}(\theta-\theta_{\underline{\Omega}})\boldsymbol{g} \end{aligned} \tag{4.81}$$

The coefficients in (4.81) are functions of the invariants and temperature θ in the known configuration $\underline{\Omega}$. This constitutive theory is linear in the components of ${}_d\boldsymbol{\sigma}^{[0]}$, $\boldsymbol{\varepsilon}_{[0]}$, and $\boldsymbol{\varepsilon}_{[1]}$, but nonlinear in $\boldsymbol{g}$. In this constitutive theory, the terms containing products of ${}_d\boldsymbol{\sigma}^{[0]}$, $\boldsymbol{\varepsilon}_{[0]}$, and $\boldsymbol{\varepsilon}_{[1]}$ tensors are neglected. We discuss some other special forms of the rate constitutive theories for $\boldsymbol{q}$ based on (4.79) and (4.81) in the following remarks.

Remarks. (1) If we consider a constitutive theory for $\boldsymbol{q}$ in which we assume that ${}_d\boldsymbol{\sigma}^{[0]}$, $\boldsymbol{\varepsilon}_{[0]}$, and $\boldsymbol{\varepsilon}_{[1]}$ do not influence $\boldsymbol{q}$, then the terms containing ${}_d\boldsymbol{\sigma}^{[0]}$, $\boldsymbol{\varepsilon}_{[0]}$, and $\boldsymbol{\varepsilon}_{[1]}$ can be deleted in (4.81), and we obtain

$$\boldsymbol{q} = -\big(\widetilde{a} + \underset{\sim}{a}\boldsymbol{g}\cdot\boldsymbol{g}\big)\boldsymbol{g} - \underset{\sim}{f}(\theta - \theta_{\underline{\Omega}})\boldsymbol{g} \tag{4.82}$$

which is the same as the constitutive theory for thermoelastic solids and thermoviscoelastic solids without memory in references [62, 65] under the same assumptions. In this case, the material coefficients $\widetilde{a}$, $\underset{\sim}{a}$, and $\underset{\sim}{f}$ are functions of $\big(\boldsymbol{g}\cdot\boldsymbol{g}\big)_{\underline{\Omega}}$ and $\theta|_{\underline{\Omega}}$.

(2) If we consider constitutive theories of orders $(1, n)$ and $(1, 1)$ that are linear in $\boldsymbol{g}$, then (4.79) and (4.81) reduce to

$$\begin{aligned} \boldsymbol{q} = -\widetilde{a} - \widetilde{b}\,{}_d\boldsymbol{\sigma}^{[0]}\boldsymbol{g} - \sum_{i=0}^{n}\widetilde{c}_i\boldsymbol{\varepsilon}_{[i]}\boldsymbol{g} \\ - \underset{\sim}{b}\big(\operatorname{tr}\,{}_d\boldsymbol{\sigma}^{[0]}\big)\boldsymbol{g} - \sum_{i=1}^{n}\underset{\sim}{c}_i\big(\operatorname{tr}\boldsymbol{\varepsilon}_{[i]}\big)\boldsymbol{g} \\ - \underset{\sim}{f}(\theta - \theta_{\underline{\Omega}})\boldsymbol{g} \end{aligned} \tag{4.83}$$

$$\begin{aligned} \boldsymbol{q} = {} & -\big(\widetilde{a}\boldsymbol{I} + \widetilde{b}\,{}_d\boldsymbol{\sigma}^{[0]} + \widetilde{c}_0\boldsymbol{\varepsilon}_{[0]} + \widetilde{c}_1\boldsymbol{\varepsilon}_{[1]}\big)\boldsymbol{g} \\ & - \big(\underset{\sim}{b}\operatorname{tr}\,{}_d\boldsymbol{\sigma}^{[0]} + \underset{\sim}{c}_0\operatorname{tr}\boldsymbol{\varepsilon}_{[0]} + \underset{\sim}{c}_1\operatorname{tr}\boldsymbol{\varepsilon}_{[1]}\big)\boldsymbol{g} - \underset{\sim}{f}(\theta - \theta_{\underline{\Omega}})\boldsymbol{g} \end{aligned} \tag{4.84}$$

(3) Simplified theory for $\boldsymbol{q}$ can also be derived using reduced argument tensors for $\boldsymbol{q}$. For example, if we consider

$$\boldsymbol{q} = \boldsymbol{q}(\boldsymbol{g}, \theta) \tag{4.85}$$

then the theory of generators and invariants yields constitutive theory (4.82) for $\boldsymbol{q}$.

(4) If we strictly use the entropy inequality with the assumption that

$$\frac{|J| q_i g_i}{\theta} \leq 0 \text{ or } q_i g_i \leq 0 \tag{4.86}$$

and if we assume that $\boldsymbol{q}$ is proportional to $-\boldsymbol{g}$, *i.e.*, linear in $\boldsymbol{g}$, then we can derive the Fourier heat conduction law for $\boldsymbol{q}$

$$q_i = -k_{ij}(\theta) g_j \tag{4.87}$$

4.6 Numerical studies

In this section, we present some numerical studies using the constitutive theory for stress tensor presented in this work. We consider the 1-d case with infinitesimal deformation and strain. We assume the material to be incompressible. For this case, the momentum and constitutive equations (in the absence of body forces) in the x_1-coordinate direction are

$$\rho_0 \frac{\partial^2 u_{x_1}}{\partial t^2} - \frac{\partial p}{\partial x_1} + \frac{\partial ({}_d\sigma_{x_1 x_1})}{\partial x_1} = 0 \tag{4.88}$$

$${}_d\sigma_{x_1 x_1} + \lambda \frac{\partial ({}_d\sigma_{x_1 x_1})}{\partial t} = e_1 \frac{\partial u_{x_1}}{\partial x_1} + n_1 \frac{\partial}{\partial t} \left(\frac{\partial u_{x_1}}{\partial x_1} \right) \tag{4.89}$$

For the incompressible case

$$p = -\frac{1}{2} \, {}_d\sigma_{x_1 x_1}$$

Hence, (4.88) in this case can be written as

$$\rho_0 \frac{\partial^2 u_{x_1}}{\partial t^2} + \frac{3}{2} \frac{\partial ({}_d\sigma_{x_1 x_1})}{\partial x_1} = 0 \tag{4.90}$$

Equations (4.89) and (4.90) constitute the mathematical model for the 1-d case used in the numerical studies presented in this section. In this mathematical model, ${}_d\sigma_{x_1 x_1}$ must be maintained as a dependent variable as the substitution of ${}_d\sigma_{x_1x_1}$ from (4.89) into (4.90) is not possible.

The mathematical model for incompressible thermoviscoelastic solids without memory for the 1-d case derived in reference [62] consists of

$$\rho_0 \frac{\partial^2 u_{x_1}}{\partial t^2} + \frac{3}{2} \frac{\partial ({}_d\sigma_{x_1x_1})}{\partial x_1} = 0 \tag{4.91}$$

$${}_d\sigma_{x_1x_1} = \widetilde{a}_{11} \frac{\partial u_{x_1}}{\partial x_1} + \widetilde{b}_{11} \frac{\partial}{\partial t} \left(\frac{\partial u_{x_1}}{\partial x_1} \right) \tag{4.92}$$

First we derive dimensionless forms of (4.89), (4.90), (4.91), and (4.92). We rewrite these by introducing '$\widehat{\ }$' (hat) on all quantities signifying that all quantities have their usual dimensions or units.

$$\widehat{\rho}_0 \frac{\partial^2 \widehat{u}_{x_1}}{\partial \widehat{t}^2} + \frac{3}{2} \frac{\partial ({}_d\widehat{\sigma}_{x_1x_1})}{\partial \widehat{x}_1} = 0 \tag{4.93}$$

$${}_d\widehat{\sigma}_{x_1x_1} + \widehat{\lambda} \frac{\partial ({}_d\widehat{\sigma}_{x_1x_1})}{\partial \widehat{t}} = \widehat{e}_1 \frac{\partial \widehat{u}_{x_1}}{\partial \widehat{x}_1} + \widehat{n}_1 \frac{\partial}{\partial \widehat{t}} \left(\frac{\partial \widehat{u}_{x_1}}{\partial \widehat{x}_1} \right) \tag{4.94}$$

$$\widehat{\rho}_0 \frac{\partial^2 \widehat{u}_{x_1}}{\partial \widehat{t}^2} + \frac{3}{2} \frac{\partial ({}_d\widehat{\sigma}_{x_1x_1})}{\partial \widehat{x}_1} = 0 \tag{4.95}$$

$${}_d\widehat{\sigma}_{x_1x_1} = \widehat{\widetilde{a}}_{11} \frac{\partial \widehat{u}_{x_1}}{\partial \widehat{x}_1} + \widehat{\widetilde{b}}_{11} \frac{\partial}{\partial \widehat{t}} \left(\frac{\partial \widehat{u}_{x_1}}{\partial \widehat{x}_1} \right) \tag{4.96}$$

We choose reference quantities with subscript zero or 'ref' and define dimensionless variables using these

$$\begin{gathered} \rho_0 = \frac{\widehat{\rho}_0}{(\rho_0)_{\text{ref}}}, \quad u_{x_1} = \frac{\widehat{u}_{x_1}}{u_0}, \quad v_{x_1} = \frac{\widehat{v}_{x_1}}{v_0}, \\ x_1 = \frac{\widehat{x}_1}{L_0}, \quad t = \frac{\widehat{t}}{t_0}, \quad {}_d\sigma_{x_1x_1} = \frac{{}_d\widehat{\sigma}_{x_1x_1}}{\tau_0} \end{gathered} \tag{4.97}$$

L_0 is a reference length, hence $u_0 = L_0$ and if we choose v_0 as a reference velocity (generally the speed of sound using reference quantities, *i.e.*, the reference speed of sound), then $t_0 = \frac{u_0}{v_0} = \frac{L_0}{v_0}$. τ_0 is a reference stress. We choose $\tau_0 = (\rho_0)_{\text{ref}} v_0^2$, the characteristic kinetic energy (which has the same dimension as stress). The dimensionless forms of (4.93) to (4.96) become

$$\rho_0 \frac{\partial^2 u_{x_1}}{\partial t^2} + \frac{3}{2}\frac{\partial(\,{}_d\sigma_{x_1x_1})}{\partial x_1} = 0 \tag{4.98}$$

$${}_d\sigma_{x_1x_1} + \text{De}\,\frac{\partial(\,{}_d\sigma_{x_1x_1})}{\partial t} = e_1^d \frac{\partial u_{x_1}}{\partial x_1} + n_1^d \frac{\partial}{\partial t}\left(\frac{\partial u_{x_1}}{\partial x_1}\right) \tag{4.99}$$

$$\rho_0 \frac{\partial^2 u_{x_1}}{\partial t^2} + \frac{3}{2}\frac{\partial(\,{}_d\sigma_{x_1x_1})}{\partial x_1} = 0 \tag{4.100}$$

$${}_d\sigma_{x_1x_1} = a_{11}^d \frac{\partial u_{x_1}}{\partial x_1} + b_{11}^d \frac{\partial}{\partial t}\left(\frac{\partial u_{x_1}}{\partial x_1}\right) \tag{4.101}$$

where

$$\begin{gathered} \text{De} = \frac{\widehat{\lambda} v_0}{L_0}; \quad e_1^d = \frac{\widehat{e}_1}{\tau_0}; \quad n_1^d = \frac{\widehat{n}_1}{(\rho_0)_{\text{ref}} v_0 L_0} \\ a_{11}^d = \frac{\widehat{\widehat{a}}_{11}}{\tau_0}; \quad b_{11}^d = \frac{\widehat{\widehat{b}}_{11}}{(\rho_0)_{\text{ref}} v_0 L_0} \end{gathered} \tag{4.102}$$

Clearly $e_1^d = a_{11}^d$ and $n_1^d = b_{11}^d$, hence a_{11}^d and b_{11}^d in (4.101) can be replaced by e_1^d and n_1^d. De is called the Deborah number. By substituting (4.101) into (4.100), we can obtain a single PDE in u_{x_1}

$$\rho_0 \frac{\partial^2 u_{x_1}}{\partial t^2} + c_1^d \frac{\partial^2 u_{x_1}}{\partial x_1^2} + c_2^d \frac{\partial}{\partial t}\left(\frac{\partial^2 u_{x_1}}{\partial x_2^2}\right) \tag{4.103}$$

in which

$$c_1^d = \frac{3}{2} e_1^d;\; c_2^d = \frac{3}{2} n_1^d \quad \text{or} \quad e_1^2 = \frac{2}{3} c_1^d;\; n_1^d = \frac{2}{3} c_2^d \tag{4.104}$$

In reference [62], numerical studies were presented for incompressible thermoviscoelastic solids without memory using (4.103) with different values of c_1^d and c_2^d. In the numerical studies presented here, we consider the mathematical model given by (4.98) and (4.99), in which we choose e_1^d and n_1^d using (4.104) with the equivalent values of c_1^d and c_2^d as used in reference [62] so that these studies can be compared with those in reference [62] to illustrate the influence of rheology.

In the numerical studies we consider an axial rod of dimensionless length one unit and choose $(\rho_0)_{\text{ref}} = \widehat{\rho}_0$ so that $\rho_0 = 1$. The spatial domain $[0, 1]$ for in increment of

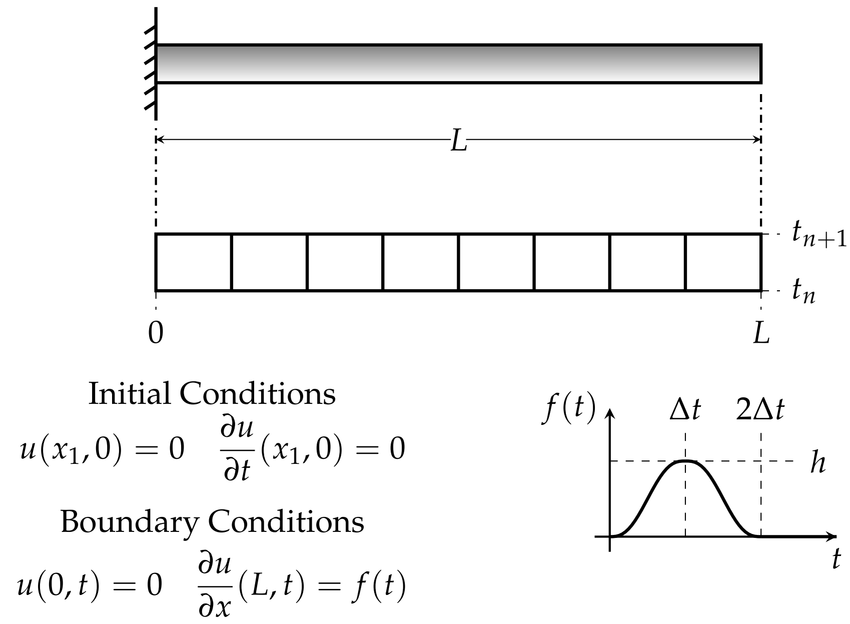

Figure 4.1: Schematic of the model problem

time $\Delta t = t_{n+1} - t_n$, *i.e.*, the n^{th} space-time strip $[0, 1] \times [t_n, t_{n+1}]$ is discretized using a uniform mesh of eight space-time p-version finite elements with higher order global differentiability. Figure 4.1 shows details of the space-time domain for an increment of time $\Delta t = t_{n+1} - t_n$, boundary conditions, as well as initial conditions. The left end of the rod is clamped (impermeable boundary) and the right end is subjected to a continuous and differentiable piecewise-cubic strain distribution over a time period of $2\Delta t$. For $t \geq 2\Delta t$, the applied strain at the right end of the rod is zero. We choose p-level of 9 in space and time, and the local approximation is of class $C^{11}(\bar{\Omega}_{x_1 t})$, *i.e.*, C^1 in space and time. The finite element formulation used for computing numerical solutions is based on space-time least squares process constructed using residual functionals. The resulting computational processes are unconditionally stable. Evolutions are computed using a space-time strip with time marching [53–57]. For the choice of local approximation $C^{11}(\bar{\Omega}_{x_1 t})$, the integrals in the finite element processes are Lebesgue, but due to the smoothness of the evolution, for the eight-element discretization with p-level of 9, the residual functionals are on the order of $O(10^{-8})$ for all space-time strips, confirming good accuracy of the evolution.

Figures 4.2 – 4.5 show evolution of stress and strain for De = 0.0, 0.03, and 0.04 for $0 \leq t \leq 2.0$. When De $= 0.0$, we use mathematical model (4.103) for computations as for this case, (4.89) loses its time term and hence is not a valid differential constitutive model. Figure 4.2 shows the evolution of stress and strain for $0 \leq t \leq 0.3$. The evolutions of stress and strain for $0.3 \leq t \leq 1.0$, $1.2 \leq t \leq 1.7$ and $1.8 \leq t \leq 2.0$ are shown in figures 4.3, 4.4, and 4.5. Numerical values of the material coefficients are also shown in the captions of the figures.

When De $= 0.0$, the behavior is thermoviscoelastic but without memory. As time elapses, the stress and strain waves experience base elongation and amplitude decay due to dissipation as shown in the figures. At time $t = 2.0$, waves show almost complete amplitude decay. For nonzero Deborah number, the stress and strain amplitude are higher than for De $= 0.0$ throughout the evolution, due to rheology. Increasing Deborah numbers produce increasing values of stress and strain during evolution. Peak stress and strain values for De $= 0.04$ are higher than those for De $= 0.03$ throughout the evolution.

4.7 Summary

In this work we have presented ordered rate constitutive theories in Lagrangian description for compressible and incompressible thermoviscoelastic solids with memory. In the derivation of the constitutive theories, the material derivative of order m of the deviatoric stress tensor and heat vector are functions of the temperature, temperature gradient, time derivatives of the conjugate strain tensor up to any desired order n and the material derivative of up to order $m-1$ of the stress tensor. The thermoviscoelastic solids described by these constitutive theories are called ordered thermoviscoelastic solids of orders (m, n). It is shown that such solids have fading memory.

The derivations of the constitutive theories are presented beginning with the entropy inequality expressed in terms of Helmholtz free energy density Φ. The Jacobian of deformation $\boldsymbol{J}$, its material derivatives, temperature, temperature gradient $\boldsymbol{g}$, and material

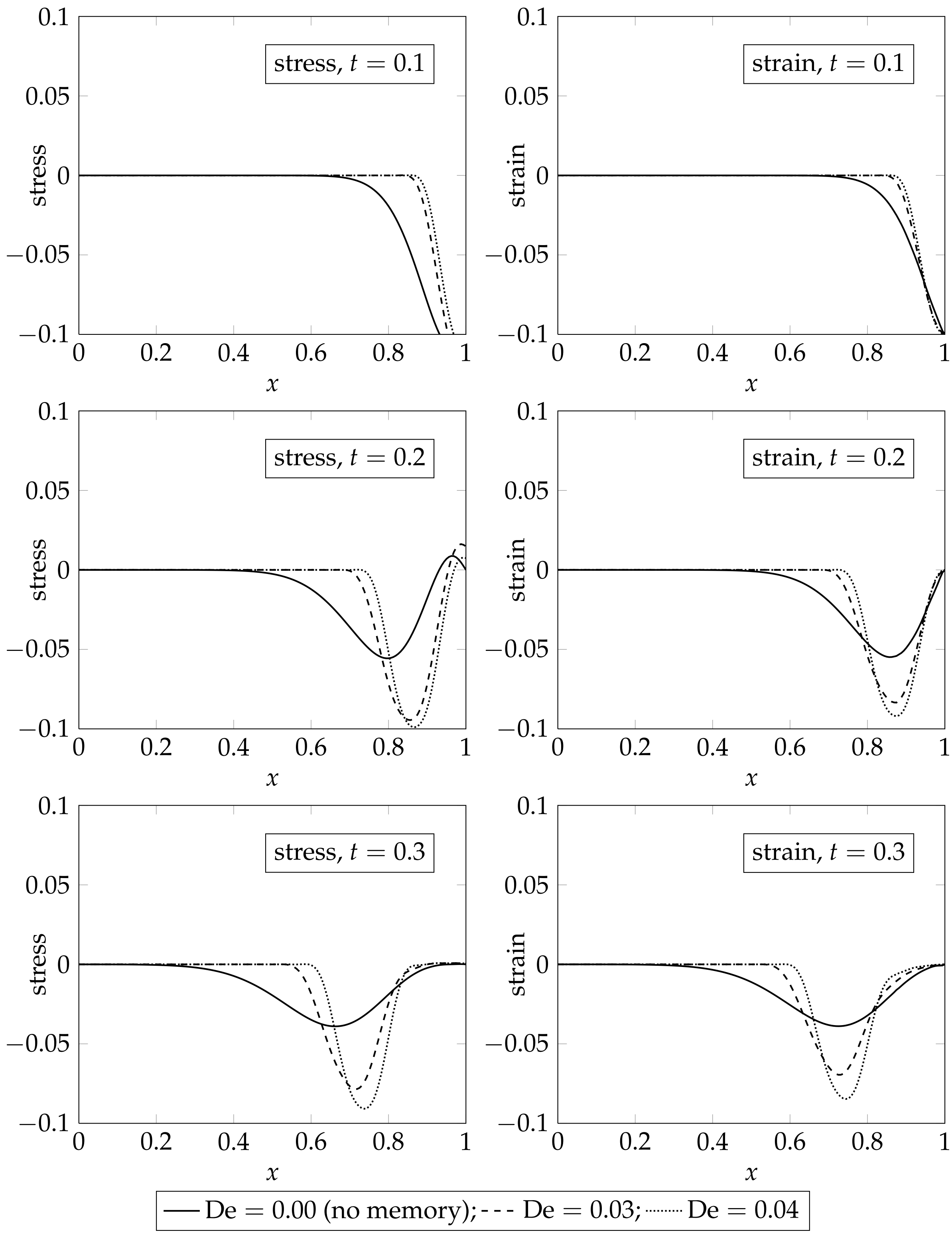

Figure 4.2: Evolutions of stress and strain for $e_1^d = 1.0$, $\eta_1^d = 0.05$ (with memory); $c_1^d = \frac{3}{2}e_1^d = 1.5$, $c_2^d = \frac{3}{2}\eta_1^d = 0.075$ (no memory)

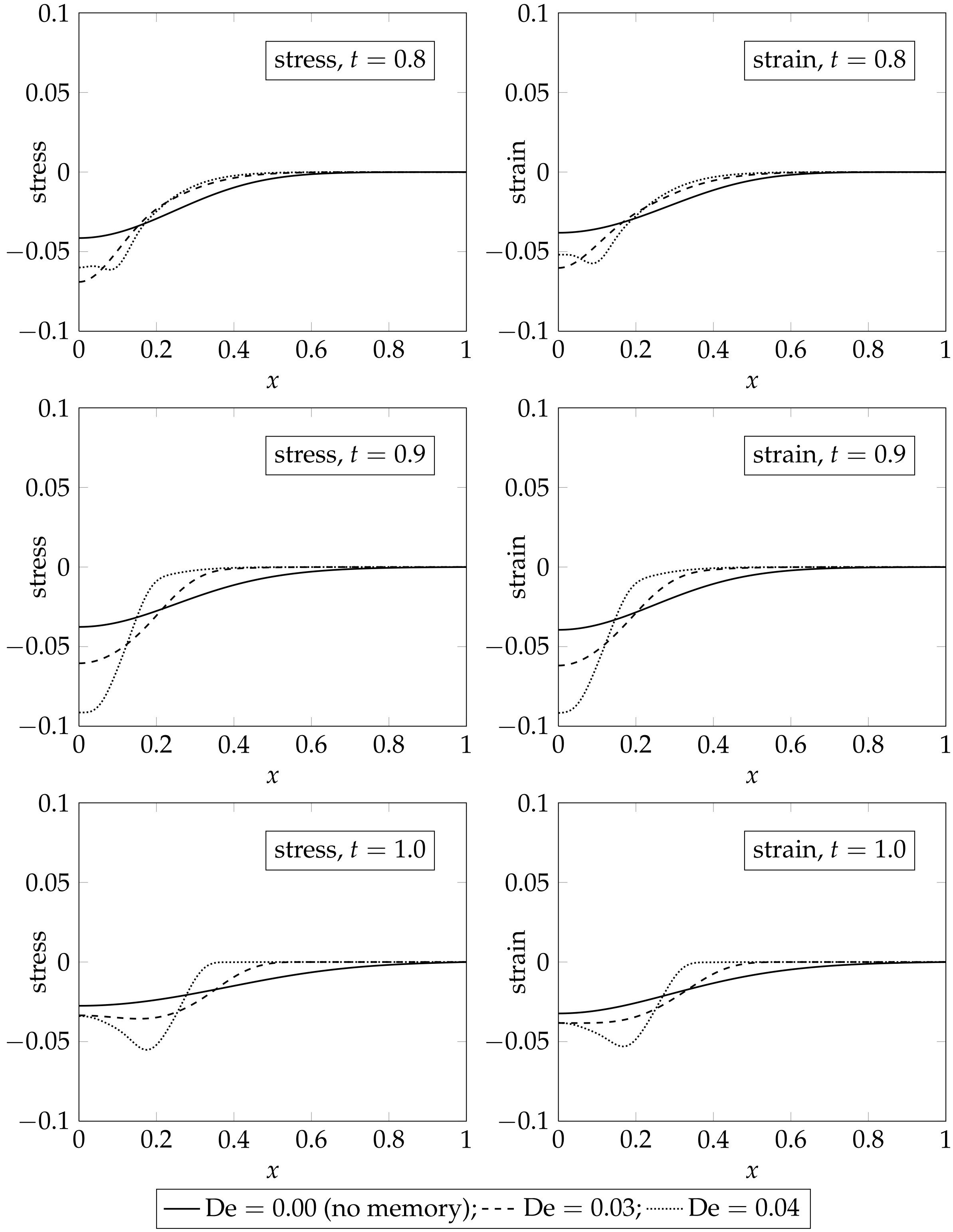

Figure 4.3: Evolutions of stress and strain for $e_1^d = 1.0$, $\eta_1^d = 0.05$ (with memory); $c_1^d = \frac{3}{2}e_1^d = 1.5$, $c_2^d = \frac{3}{2}\eta_1^d = 0.075$ (no memory)

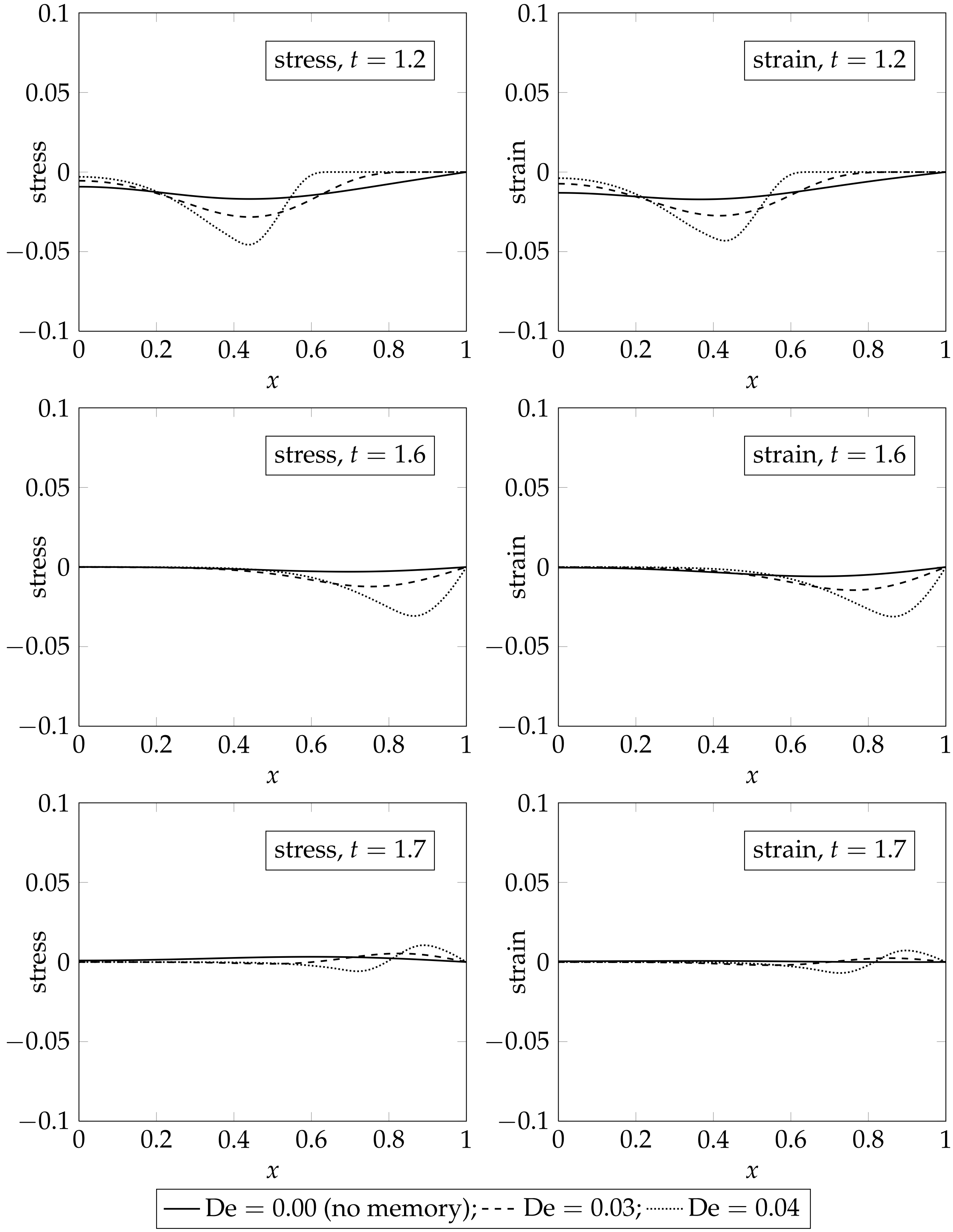

Figure 4.4: Evolutions of stress and strain for $e_1^d = 1.0$, $\eta_1^d = 0.05$ (with memory); $c_1^d = \frac{3}{2}e_1^d = 1.5$, $c_2^d = \frac{3}{2}\eta_1^d = 0.075$ (no memory)

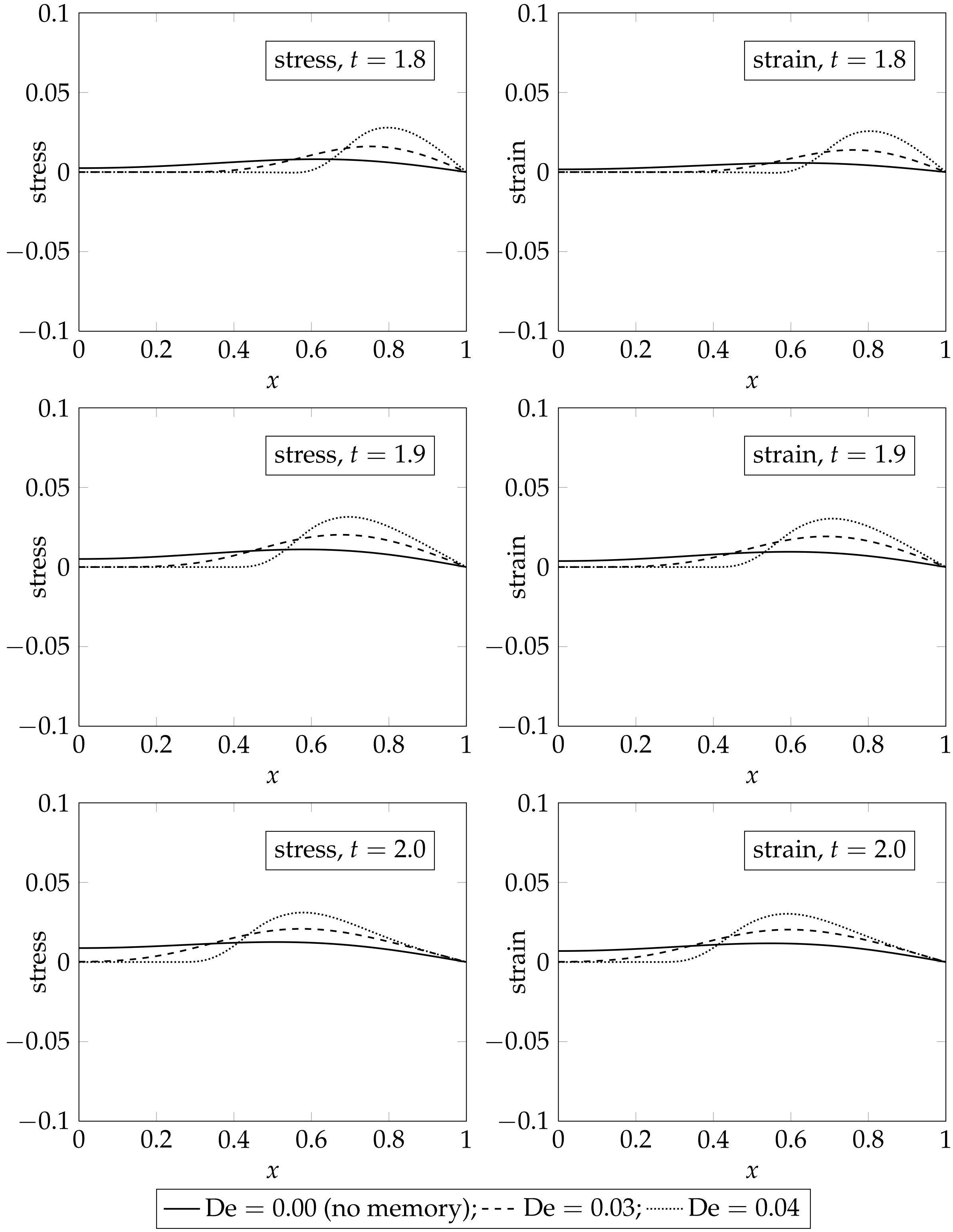

Figure 4.5: Evolutions of stress and strain for $e_1^d = 1.0$, $\eta_1^d = 0.05$ (with memory); $c_1^d = \frac{3}{2}e_1^d = 1.5$, $c_2^d = \frac{3}{2}\eta_1^d = 0.075$ (no memory)

derivatives of σ^* up to orders $m-1$ are all considered as arguments of Φ, $\boldsymbol{q}$, η, and $\sigma^{*[m]}$ at the onset of the derivation. Using $\dot{\Phi}$ in the entropy inequality expressed through arguments of Φ and its substitution in the entropy inequality provides much-needed reduction of the arguments of Φ as well as eliminating η as a dependent variable in the constitutive theory. In the end, we only have Φ, $\sigma^{*[m]}$, and $\boldsymbol{q}$ as dependent variables in the constitutive theory. Using the axiom of frame invariance, $\boldsymbol{J}_{[i]}$; $i = 0, 1, \ldots, n$ are replaced by $\boldsymbol{\varepsilon}_{[i]}$; $i = 0, 1, \ldots, n$ and $\sigma^{*[j]}$; $j = 0, 1, \ldots, m$ by $\sigma^{[j]}$; $j = 0, 1, \ldots, m$, conjugate to $\boldsymbol{\varepsilon}_{[i]}$; $i = 0, 1, \ldots, n$. At this stage, the conditions resulting from the entropy inequality do not provide any mechanism to derive constitutive theory for the stress tensor. A decomposition of the stress tensor into equilibrium and deviatoric stress tensors enables determination of the constitutive theories for the equilibrium stress tensor for compressible and incompressible cases. The constitutive theory for the deviatoric stress tensor is derived using the theory of generators and invariants. This approach is based on axioms and principles of continuum mechanics. However, we keep in mind that these constitutive theories must satisfy the inequalities resulting from the second law of thermodynamics.

The constitutive theories for heat vector $\boldsymbol{q}$ are derived: (i) strictly using conditions resulting from the entropy inequality; (ii) using the theory of generators and invariants with admissible argument tensors that are consistent with the stress tensor as well as the theories in which simplifying assumptions are employed which yield much-simplified theories. It is shown that the rate theories presented here describe thermoviscoelastic solids with memory. Mechanisms of dissipation and memory are demonstrated and discussed, and derivation of memory modulus is presented. It is shown that simplified forms of the general theories presented here result in constitutive models that may resemble currently-used constitutive models but are not the same. The work presented here is not to be viewed as extensions of the current constitutive models, rather, it is a general framework for rate constitutive theories for thermoviscoelastic solids with memory based on the physics and derivations that are consistent within the framework of continuum mechanics and thermodynamics. The purpose of the simplified theories presented in the paper is to illustrate possible simple theories within the consistent framework presented here.

The derivation of the memory modulus for the 1-d case shows that the memory modulus is dependent on strain as well as strain rates of all orders. When $(\varepsilon)_{x_1x_1} = 0$ as in fluids and we only consider $\frac{\partial(\varepsilon)_{x_1x_1}}{\partial t}$ as the strain rate, the memory modulus reduces similar to the well-known memory modulus for Maxwell fluid. Numerical examples are presented for infinitesimal deformation and strain to demonstrate the influence of rheology on the stress and strain fields during evolution for progressively increasing Deborah number.

Chapter 5

Summary and conclusions

Ordered rate constitutive theories in Lagrangian description for thermoelastic solids, thermoviscoelastic solids without memory, and thermoviscoelastic solids with memory have been presented. These theories are derived using the entropy inequality in terms of the Helmholtz free energy density Φ. It is shown that the material derivative of order m of the deviatoric stress tensor is a function of the temperature θ, temperature gradient $\boldsymbol{g}$, material derivatives of up to order n of the conjugate strain tensor, and material derivatives of up to order $m-1$ of the deviatoric stress tensor. These theories are known as rate constitutive theories of orders (m, n). It is further shown that thermoelastic solids are described by rate constitutive theories of order zero (*i.e.*, $m = 0, n = 0$), thermoviscoelastic solids without memory are described by rate constitutive theories of order n (*i.e.*, $m = 0$), and that thermoviscoelastic solids with memory are described by rate constitutive theories of order (m, n), where $m \geq 1$.

The axiom of admissibility requires that the constitutive theories satisfy all conservation equations. Conservation of mass, balance of momenta, and conservation of energy only require the existence of a stress tensor and heat vector, but they provide no mechanism for derivation. The entropy inequality, therefore, must form the basis for deriving constitutive theories. Derivations are presented by beginning with the entropy inequality expressed in terms of the Helmholtz free energy density. At the outset of the derivation, the Helmholtz free energy density Φ, material derivative of order m of the first Piola-Kirchhoff stress tensor $\boldsymbol{\sigma}^{*[m]}$, heat vector $\boldsymbol{q}$, and entropy density η are considered as possible dependent variables in the theories, with the Jacobian of deformation $\boldsymbol{J}$ and its material derivatives, temperature θ, temperature gradient $\boldsymbol{g}$, and $\boldsymbol{\sigma}^{*[j]}$; $j = 0, 1, \ldots, m-1$ as possible argument tensors.

The conditions resulting from the entropy inequality provide the following conclusions:

(i) The entropy density η is deterministic from the Helmholtz free energy density and temperature, and as such, it cannot be a dependent variable in the constitutive theories.

(ii) The Helmholtz free energy density depends only on the Jacobian of deformation and the temperature.

(iii) Helmholtz free energy density, stress tensor, and heat vector are identified as dependent variables in the constitutive theories.

(iv) At this point, the entropy inequality provides no further mechanism to derive constitutive theories for $\boldsymbol{\sigma}^*$ and $\boldsymbol{q}$.

To proceed, the stress tensor is decomposed into equilibrium stress ${}_e\boldsymbol{\sigma}^*$ and deviatoric stress ${}_d\boldsymbol{\sigma}^*$. The equilibrium stress is dependent only on the Jacobian of deformation and the temperature and represents thermodynamic pressure in the case of compressible material and mechanical pressure in the incompressible case. Substituting this decomposition into the entropy inequality allows the inequality to be expressed using only the heat vector and deviatoric stress tensor as dependent variables. This inequality is satisfied if the inner product of the heat vector and the temperature gradient vector is nonpositive and the rate of work expended due to the deviatoric stress is positive. The entropy inequality, however, provides no further guidance in deriving the constitutive theories.

The selection of $\boldsymbol{\sigma}^*$ as a dependent variable and $\boldsymbol{J}$ as an argument does not satisfy the continuum mechanics axiom of frame invariance. This axiom is satisfied by replacing these with the second Piola-Kirchhoff stress tensor $\boldsymbol{\sigma}^{[0]}$ and Green's strain $\boldsymbol{\varepsilon}_{[0]}$. Derivations are given that show equivalent conditions based on this selection of variables,

To derive constitutive theories for deviatoric second Piola-Kirchhoff stress ${}_{d}\boldsymbol{\sigma}^{[0]}$ and heat vector $\boldsymbol{q}$, the theory of generators and invariants is used. This provides a framework for deriving all possible constitutive theories for a given selection of arguments. In the following, we note some conclusions from the work presented here

(1) It is shown here that all constitutive theories are rate constitutive theories of order (m, n). Restrictions on the order of these theories determines the type of material described. Theories with order $m > 0$ describe thermoviscoelastic solids with memory. Theories with $m = 0$ but $n > 0$ describe thermoviscoelastic solids without memory. Theories with $m = 0$ and $n = 0$ describe thermoelastic solids.

(2) For rate constitutive theories of order (m, n), the condition $m \leq n$ is required [35].

(3) For any set of argument tensors, all constitutive theories are linear combinations of the combined generators of these tensors. The coefficients of these generators are scalar-valued functions of the combined invariants of the argument tensors. A complete list of generators and invariants is given in Appendix A.

(4) Because the coefficients of the generators can be, in general, *any* continuous scalar-valued function of the invariants, theories derived using this technique are not immediately useful. A Taylor series of each coefficient in the invariants and temperature is constructed about a known configuration and is truncated after the linear terms. This allows determination of material coefficients for any given theory.

(5) The number of material coefficients for a theory grows drastically with each added argument tensor, due to the wider array of generators, invariants, and their associated combinations. For example, a constitutive theory for ${}_{d}\boldsymbol{\sigma}^{[0]}$ of order $(0, 0)$, *i.e.*, a

thermoelastic solid, with stress dependent on strain and temperature only requires 14 material coefficients. To compare, a constitutive theory for ${}_d\sigma^{[0]}$ of order $(0, 1)$, with stress dependent on strain, strain rate, and temperature requires 95 material coefficients. Adding the temperature gradient as an argument tensor increases the number of material coefficients to 233.

(6) Constitutive theories for the heat vector that include the effects of strain and stress on heat conduction are easily constructed using the methodology presented here. These theories are necessarily far more complex than the Fourier heat conduction law, due to the explosive growth in the number of material coefficients caused by the addition of argument tensors as described above.

(7) By selectively neglecting coefficients, theories that resemble commonly used models such as Kelvin-Voigt and Zener can be created. The common models such as Kelvin-Voigt and Zener are phenomenological and one-dimensional in nature. As such, they do not distinguish between deviatoric and total stress, nor do they have a clear mechanism for extension into higher dimensions, finite strain and deformation, or compressible material. The theories presented here, however, do not suffer from these deficiencies. Rather, they are complete and valid theories that satisfy all axioms and principles of continuum mechanics by construction.

(8) For thermoelastic solids, comparisons are made between the results of the theories based on the framework presented here and other common methods of deriving constitutive equations for thermoelastic solids. It is shown that the constitutive model obtained by expanding Φ in a Taylor series in $\varepsilon_{[0]}$ about a known configuration is not the same as that obtained from the theories presented here. Since the Taylor series expansion in $\varepsilon_{[0]}$ is not based on continuum mechanics or thermodynamics principles, the framework presented here is meritorious.

(9) Numerical studies are presented illustrating the effect of viscous dissipation for ther-

moviscoelastic solids without memory. These studies are compared with results from a traditional velocity-based structural damping simulation to illustrate the difference in the two dissipation mechanism.

(10) Numerical studies are presented for thermoviscoelastic solids with memory illustrating the memory effect due to the presence of ${}_{d}\sigma^{[1]}$. These results are compared with equivalent studies for thermoviscoelastic solids without memory to clearly illustrate the influence of stress rates in the constitutive theories.

The work presented here provides general and unified ordered rate constitutive theories for thermoelastic solids, thermoviscoelastic solids without memory, and thermoviscoelastic solids with memory based on the axioms and principles of continuum mechanics and thermodynamics. These theories provide a thermodynamically sound framework to derive specific constitutive models based on a chosen set of arguments. From these theories, constitutive models that resemble commonly used phenomenological models such as Kelvin-Voigt and Zener have been derived and presented.

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Appendix A

Combined generators and invariants

The following presents a complete list of invariants, generators that are tensors of rank one, generators that are symmetric tensors of rank two, and generators that are skew-symmetric tensors of rank two. References for these are given in the introduction.

For these lists, let $\boldsymbol{v}_\alpha, \boldsymbol{v}_\beta$; $\alpha, \beta = 1, 2, \ldots, P$; $\alpha < \beta$ be tensors of rank one, *i.e.*, vectors, $\boldsymbol{A}_i, \boldsymbol{A}_j, \boldsymbol{A}_k$; $i, j, k = 1, 2, \ldots, N$; $i < j < k$ be symmetric tensors of rank two, and $\boldsymbol{W}_p, \boldsymbol{W}_q, \boldsymbol{W}_r$; $p, q, r = 1, 2, \ldots, M$; $p < q < r$ be skew-symmetric tensors of rank two.

Every scalar-valued isotropic function can be written as a function of the invariants given in the following list.

$$
\begin{aligned}
&\boldsymbol{v}_\alpha \cdot \boldsymbol{v}_\alpha, \quad \boldsymbol{v}_\alpha \cdot \boldsymbol{v}_\beta, \\
&\operatorname{tr} \boldsymbol{A}_i, \quad \operatorname{tr} \boldsymbol{A}_i^2, \quad \operatorname{tr} \boldsymbol{A}_i^3, \quad \operatorname{tr} \boldsymbol{A}_i \boldsymbol{A}_j, \quad \operatorname{tr} \boldsymbol{A}_i^2 \boldsymbol{A}_j, \quad \operatorname{tr} \boldsymbol{A}_i \boldsymbol{A}_j^2, \quad \operatorname{tr} \boldsymbol{A}_i^2 \boldsymbol{A}_j^2, \quad \operatorname{tr} \boldsymbol{A}_i \boldsymbol{A}_j \boldsymbol{A}_k, \\
&\operatorname{tr} \boldsymbol{W}_p^2, \quad \operatorname{tr} \boldsymbol{W}_p \boldsymbol{W}_q, \quad \operatorname{tr} \boldsymbol{W}_p \boldsymbol{W}_q \boldsymbol{W}_r, \\
&\boldsymbol{v}_\alpha \cdot \boldsymbol{A}_i \boldsymbol{v}_\alpha, \quad \boldsymbol{v}_\alpha \cdot \boldsymbol{A}_i^2 \boldsymbol{v}_\alpha, \quad \boldsymbol{v}_\alpha \cdot \boldsymbol{A}_i \boldsymbol{A}_j \boldsymbol{v}_\alpha, \\
&\boldsymbol{v}_\alpha \cdot \boldsymbol{A}_i \boldsymbol{v}_\beta, \quad \boldsymbol{v}_\alpha \cdot \boldsymbol{A}_i^2 \boldsymbol{v}_\beta, \quad \boldsymbol{v}_\alpha \cdot (\boldsymbol{A}_i \boldsymbol{A}_j - \boldsymbol{A}_j \boldsymbol{A}_i) \boldsymbol{v}_\beta, \\
&\boldsymbol{v}_\alpha \cdot \boldsymbol{W}_p^2 \boldsymbol{v}_\alpha, \quad \boldsymbol{v}_\alpha \cdot \boldsymbol{W}_p \boldsymbol{W}_q \boldsymbol{v}_\alpha, \quad \boldsymbol{v}_\alpha \cdot \boldsymbol{W}_p^2 \boldsymbol{W}_q \boldsymbol{v}_\alpha, \quad \boldsymbol{v}_\alpha \cdot \boldsymbol{W}_p \boldsymbol{W}_q^2 \boldsymbol{v}_\alpha, \\
&\boldsymbol{v}_\alpha \cdot \boldsymbol{W}_p \boldsymbol{v}_\beta, \quad \boldsymbol{v}_\alpha \cdot \boldsymbol{W}_p^2 \boldsymbol{v}_\beta, \quad \boldsymbol{v}_\alpha \cdot (\boldsymbol{W}_p \boldsymbol{W}_q - \boldsymbol{W}_q \boldsymbol{W}_p) \boldsymbol{v}_\beta, \\
&\operatorname{tr} \boldsymbol{A}_i \boldsymbol{W}_p^2, \quad \operatorname{tr} \boldsymbol{A}_i^2 \boldsymbol{W}_p^2, \quad \operatorname{tr} \boldsymbol{A}_i^2 \boldsymbol{W}_p^2 \boldsymbol{A}_i \boldsymbol{W}_p, \quad \operatorname{tr} \boldsymbol{A}_i \boldsymbol{W}_p \boldsymbol{W}_q, \quad \operatorname{tr} \boldsymbol{A}_i \boldsymbol{W}_p \boldsymbol{W}_q^2, \quad \operatorname{tr} \boldsymbol{A}_i \boldsymbol{W}_p^2 \boldsymbol{W}_q, \\
&\operatorname{tr} \boldsymbol{A}_i \boldsymbol{A}_j \boldsymbol{W}_p, \quad \operatorname{tr} \boldsymbol{A}_i \boldsymbol{W}_p^2 \boldsymbol{A}_j \boldsymbol{W}_p, \quad \operatorname{tr} \boldsymbol{A}_i \boldsymbol{A}_j^2 \boldsymbol{W}_p, \quad \operatorname{tr} \boldsymbol{A}_i^2 \boldsymbol{A}_j \boldsymbol{W}_p, \\
&\boldsymbol{v}_\alpha \cdot \boldsymbol{A}_i \boldsymbol{W}_p \boldsymbol{v}_\alpha, \quad \boldsymbol{v}_\alpha \cdot \boldsymbol{W}_p \boldsymbol{A}_i \boldsymbol{W}_p^2 \boldsymbol{v}_\alpha, \quad \boldsymbol{v}_\alpha \cdot \boldsymbol{A}_i^2 \boldsymbol{W}_p \boldsymbol{v}_\alpha, \\
&\boldsymbol{v}_\alpha \cdot (\boldsymbol{A}_i \boldsymbol{W}_p - \boldsymbol{W}_p \boldsymbol{A}_i) \boldsymbol{v}_\beta.
\end{aligned}
\tag{A.1}
$$

Every vector-valued isotropic function can be written as a linear combination of the generators given in the following list.

$$
\begin{aligned}
&\boldsymbol{v}_\alpha, \\
&\boldsymbol{A}_i \boldsymbol{v}_\alpha, \quad \boldsymbol{A}_i^2 \boldsymbol{v}_\alpha, \quad \boldsymbol{A}_i \boldsymbol{A}_j \boldsymbol{v}_\alpha, \quad \boldsymbol{A}_j \boldsymbol{A}_i \boldsymbol{v}_\alpha, \\
&\boldsymbol{W}_p \boldsymbol{v}_\alpha, \quad \boldsymbol{W}_p^2 \boldsymbol{v}_\alpha, \quad (\boldsymbol{W}_p \boldsymbol{W}_q - \boldsymbol{W}_q \boldsymbol{W}_p) \boldsymbol{v}_\alpha, \\
&(\boldsymbol{A}_i \boldsymbol{W}_p - \boldsymbol{W}_p \boldsymbol{A}_i) \boldsymbol{v}_\alpha
\end{aligned}
\tag{A.2}
$$

where the coefficients of these generators are scalar-valued isotropic functions formed from the list (A.1).

Every symmetric tensor-valued isotropic function can be written as a linear combination of the generators given in the following list

$$
\begin{aligned}
&\boldsymbol{I}, \\
&\boldsymbol{A}_i, \quad \boldsymbol{A}_i^2, \quad \boldsymbol{A}_i\boldsymbol{A}_j + \boldsymbol{A}_j\boldsymbol{A}_i, \quad \boldsymbol{A}_i^2\boldsymbol{A}_j + \boldsymbol{A}_j\boldsymbol{A}_i^2, \quad \boldsymbol{A}_i\boldsymbol{A}_j^2 + \boldsymbol{A}_j^2\boldsymbol{A}_i, \\
&\boldsymbol{v}_\alpha \otimes \boldsymbol{v}_\alpha, \quad \boldsymbol{v}_\alpha \otimes \boldsymbol{v}_\beta + \boldsymbol{v}_\beta \otimes \boldsymbol{v}_\alpha, \\
&\boldsymbol{W}_p^2, \quad \boldsymbol{W}_p\boldsymbol{W}_q + \boldsymbol{W}_q\boldsymbol{W}_p, \quad \boldsymbol{W}_p\boldsymbol{W}_q^2 - \boldsymbol{W}_q^2\boldsymbol{W}_p, \quad \boldsymbol{W}_p^2\boldsymbol{W}_q - \boldsymbol{W}_q\boldsymbol{W}_p^2, \\
&\boldsymbol{v}_\alpha \otimes \boldsymbol{A}_i\boldsymbol{v}_\alpha + \boldsymbol{A}_i\boldsymbol{v}_\alpha \otimes \boldsymbol{v}_\alpha, \quad \boldsymbol{v}_\alpha \otimes \boldsymbol{A}_i^2\boldsymbol{v}_\alpha + \boldsymbol{A}_i^2\boldsymbol{v}_\alpha \otimes \boldsymbol{v}_\alpha, \\
&\boldsymbol{A}_i(\boldsymbol{v}_\alpha \otimes \boldsymbol{v}_\beta - \boldsymbol{v}_\beta \otimes \boldsymbol{v}_\alpha) - (\boldsymbol{v}_\alpha \otimes \boldsymbol{v}_\beta - \boldsymbol{v}_\beta \otimes \boldsymbol{v}_\alpha)\boldsymbol{A}_i, \\
&\boldsymbol{A}_i\boldsymbol{W}_p - \boldsymbol{W}_p\boldsymbol{A}_i, \quad \boldsymbol{W}_p\boldsymbol{A}_i\boldsymbol{W}_p, \quad \boldsymbol{A}_i^2\boldsymbol{W}_p - \boldsymbol{W}_p\boldsymbol{A}_i^2, \quad \boldsymbol{W}_p\boldsymbol{A}_i\boldsymbol{W}_p^2 - \boldsymbol{W}_p^2\boldsymbol{A}_i\boldsymbol{W}_p, \\
&\boldsymbol{W}_p\boldsymbol{v}_\alpha \otimes \boldsymbol{W}_p\boldsymbol{v}_\alpha, \quad \boldsymbol{v}_\alpha \otimes \boldsymbol{W}_p\boldsymbol{v}_\alpha + \boldsymbol{W}_p\boldsymbol{v}_\alpha \otimes \boldsymbol{v}_\alpha, \quad \boldsymbol{W}_p\boldsymbol{v}_\alpha \otimes \boldsymbol{W}_p^2\boldsymbol{v}_\alpha + \boldsymbol{W}_p^2\boldsymbol{v}_\alpha \otimes \boldsymbol{W}_p\boldsymbol{v}_\alpha, \\
&\boldsymbol{W}_p(\boldsymbol{v}_\alpha \otimes \boldsymbol{v}_\beta - \boldsymbol{v}_\beta \otimes \boldsymbol{v}_\alpha) + (\boldsymbol{v}_\alpha \otimes \boldsymbol{v}_\beta - \boldsymbol{v}_\beta \otimes \boldsymbol{v}_\alpha)\boldsymbol{W}_p
\end{aligned}
\tag{A.3}
$$

where the coefficients of these generators are scalar-valued isotropic functions formed from the list (A.1).

Every skew-symmetric tensor-valued isotropic function can be written as a linear combination of the generators given in the following list

$$
\begin{aligned}
&\boldsymbol{W}_p, \quad \boldsymbol{W}_p\boldsymbol{W}_q - \boldsymbol{W}_q\boldsymbol{W}_p,\\
&\boldsymbol{A}_i\boldsymbol{A}_j - \boldsymbol{A}_j\boldsymbol{A}_i, \quad \boldsymbol{A}_i^2\boldsymbol{A}_j - \boldsymbol{A}_j\boldsymbol{A}_i^2, \quad \boldsymbol{A}_i\boldsymbol{A}_j^2 - \boldsymbol{A}_j^2\boldsymbol{A}_i, \quad \boldsymbol{A}_i\boldsymbol{A}_j\boldsymbol{A}_i^2 - \boldsymbol{A}_i^2\boldsymbol{A}_j\boldsymbol{A}_i,\\
&\boldsymbol{A}_j\boldsymbol{A}_i\boldsymbol{A}_j^2 - \boldsymbol{A}_j^2\boldsymbol{A}_i\boldsymbol{A}_j, \quad \boldsymbol{A}_i\boldsymbol{A}_j\boldsymbol{A}_k + \boldsymbol{A}_j\boldsymbol{A}_k\boldsymbol{A}_i + \boldsymbol{A}_k\boldsymbol{A}_i\boldsymbol{A}_j - \boldsymbol{A}_j\boldsymbol{A}_i\boldsymbol{A}_k - \boldsymbol{A}_i\boldsymbol{A}_k\boldsymbol{A}_j - \boldsymbol{A}_k\boldsymbol{A}_j\boldsymbol{A}_i,\\
&\boldsymbol{v}_\alpha \otimes \boldsymbol{v}_\beta - \boldsymbol{v}_\beta \otimes \boldsymbol{v}_\alpha,\\
&\boldsymbol{v}_\alpha \otimes \boldsymbol{A}_i\boldsymbol{v}_\alpha - \boldsymbol{A}_i\boldsymbol{v}_\alpha \otimes \boldsymbol{v}_\alpha, \quad \boldsymbol{v}_\alpha \otimes \boldsymbol{A}_i^2\boldsymbol{v}_\alpha - \boldsymbol{A}_i^2\boldsymbol{v}_\alpha \otimes \boldsymbol{v}_\alpha,\\
&\boldsymbol{A}_i\boldsymbol{v}_\alpha \otimes \boldsymbol{A}_i^2\boldsymbol{v}_\alpha - \boldsymbol{A}_i^2\boldsymbol{v}_\alpha \otimes \boldsymbol{A}_i\boldsymbol{v}_\alpha,\\
&\boldsymbol{A}_i\boldsymbol{v}_\alpha \otimes \boldsymbol{A}_j\boldsymbol{v}_\alpha - \boldsymbol{A}_j\boldsymbol{v}_\alpha \otimes \boldsymbol{A}_i\boldsymbol{v}_\alpha\boldsymbol{v}_\alpha \otimes (\boldsymbol{A}_i\boldsymbol{A}_j - \boldsymbol{A}_j\boldsymbol{A}_i)\boldsymbol{v}_\alpha - (\boldsymbol{A}_i\boldsymbol{A}_j - \boldsymbol{A}_j\boldsymbol{A}_i)\boldsymbol{v}_\alpha \otimes \boldsymbol{v}_\alpha,\\
&\boldsymbol{A}_i(\boldsymbol{v}_\alpha \otimes \boldsymbol{v}_\beta - \boldsymbol{v}_\beta \otimes \boldsymbol{v}_\alpha) + (\boldsymbol{v}_\alpha \otimes \boldsymbol{v}_\beta - \boldsymbol{v}_\beta \otimes \boldsymbol{v}_\alpha)\boldsymbol{A}_i,\\
&\boldsymbol{A}_i\boldsymbol{W}_p + \boldsymbol{W}_p\boldsymbol{A}_i, \quad \boldsymbol{A}_i\boldsymbol{W}_p^2 - \boldsymbol{W}_p^2\boldsymbol{A}_i,\\
&\boldsymbol{v}_\alpha \otimes \boldsymbol{W}_p\boldsymbol{v}_\alpha - \boldsymbol{W}_p\boldsymbol{v}_\alpha \otimes \boldsymbol{v}_\alpha, \quad \boldsymbol{v}_\alpha \otimes \boldsymbol{W}_p^2\boldsymbol{v}_\alpha - \boldsymbol{W}_p^2\boldsymbol{v}_\alpha \otimes \boldsymbol{v}_\alpha,\\
&\boldsymbol{W}_p(\boldsymbol{v}_\alpha \otimes \boldsymbol{v}_\beta - \boldsymbol{v}_\beta \otimes \boldsymbol{v}_\alpha) - (\boldsymbol{v}_\alpha \otimes \boldsymbol{v}_\beta - \boldsymbol{v}_\beta \otimes \boldsymbol{v}_\alpha)\boldsymbol{W}_p
\end{aligned}
\tag{A.4}
$$

where the coefficients of these generators are scalar-valued isotropic functions formed form the list (A.1).